Unit 4

Vectors and matrices

Introduction

This book is largely concerned with *matrices*, i.e. rectangular arrays of numbers or other elements. Such arrays have many applications and are an essential tool in many fields of study, including physics, engineering and of course mathematics itself.

The book consists of three units. The first, Unit 4, concerns *vectors* and *matrices*. It starts with a review of vectors – a mainly geometric concept that you should have already encountered in your earlier studies. It then introduces matrices and shows how matrices that take the form of a single row or column can be used to represent a vector. Having formed this link between its two main subjects, the unit goes on to show how square matrices can be used to represent geometric *transformations*, such as stretching (dilation) or turning (rotation). From this it develops the general rules for adding and multiplying matrices, and hence provides the foundations of *matrix algebra*.

The remaining two units, Units 5 and 6, continue the discussion of matrices. Unit 5 concentrates particularly on the use of matrices in the treatment of *systems of simultaneous linear equations*, and thus leads to the subject known as *linear algebra*. Unit 6 extends this work by focusing on the treatment of *systems of linear differential equations*.

Studying these three units in sequence will teach you a great deal about matrices, and will especially emphasise the significance of *eigenvalues* and *eigenvectors*, two concepts that will be mentioned several times and which play an important part in many different applications.

Study guide

Section 1 reviews simple features of *scalar* and *vector* quantities in two and three dimensions. Much of this should be familiar to you from previous study. If so, feel free to skim the text, but make sure that you do the exercises.

Section 2 explains two important ways of forming a product from two vectors $\mathbf{a}$ and $\mathbf{b}$. The *scalar product* $\mathbf{a} \cdot \mathbf{b}$ produces a scalar quantity. The *vector product* $\mathbf{a} \times \mathbf{b}$ produces a vector, perpendicular to the plane containing $\mathbf{a}$ and $\mathbf{b}$. Each kind of product is of great utility in the physical sciences.

Section 3 introduces *matrices*. It relates vectors and matrices, and considers the use of square matrices to represent geometric *transformations* of a plane. This leads to a discussion of the *multiplication of matrices* and hence to some simple examples of *matrix algebra*.

Section 4 provides more practice in matrix multiplication and gives the general rules of matrix algebra, applicable to matrices of any size. In addition, it provides methods for calculating entities known as *determinants* and *inverses* of matrices. It also revisits the scalar and vector products of Section 2, using matrix notation, thereby emphasising the ability of matrix methods to encapsulate and simplify important results.

1 Vectors

1.1 Indicating and representing vectors

One way to distinguish *vector* quantities from *scalar* quantities is as follows.

> A **scalar** quantity is one that can be specified by a single number or by the combination of a number and a unit of measurement.
>
> A **vector** quantity is one that requires both a **magnitude** *and* a **direction** for its complete specification.

Examples of scalar quantities include the *number* $\pi = 3.1415\ldots$, a *mass* of 5 kg, a *distance* of 2.5 m, a *speed* of $3.0 \times 10^{-6}\,\mathrm{m\,s^{-1}}$, and a *temperature* of $-30°\mathrm{C}$. Those scalars that can be described by numbers alone, such as the number of peas in a pod, are called **numerical quantities**.

A common example of a vector quantity is **displacement**. The displacement from London to Brighton is (approximately) 74 km due South, and the (approximate) displacement from Milton Keynes to Oxford is 47 km South-West. Note that the specification of a displacement involves a magnitude (in this case given by a distance in km) and a direction (in this case given as a compass bearing). The magnitude of a vector quantity is not allowed to be negative. It is correct to say that the displacement from Brighton to London is 74 km North, but it is not correct to describe that displacement as −74 km South.

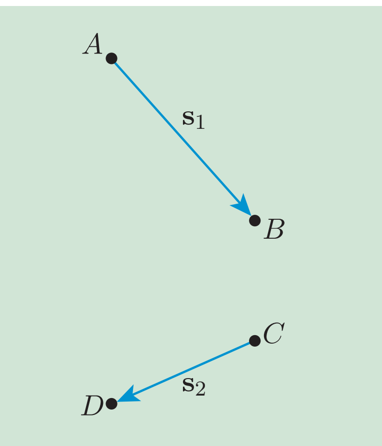


Figure 1 Displacement vectors represented by arrows may be indicated by symbols such as $\overrightarrow{AB}$ or $\mathbf{s}_1$

Figure 1 represents part of a plane that includes the points A, B, C and D. The displacement from A to B is indicated by an arrow. That is appropriate since an arrow has a magnitude (its length expressed in appropriate units) and a direction (its orientation).

A symbol that is conventionally used to represent the displacement from A to B is $\overrightarrow{AB}$. However, an even more common convention is the use of a bold letter, traditionally $\mathbf{s}$, to represent a displacement. This too can be seen in Figure 1, where the displacement $\overrightarrow{AB}$ is represented by the symbol $\mathbf{s}_1$, and the displacement $\overrightarrow{CD}$ is represented by $\mathbf{s}_2$.

Displacement, specified by a direction and a distance, is not the only vector quantity. Other examples include *velocity*, which is specified by a direction and a speed, and *force*, which is specified by a direction and a force strength. Note that distance, speed and force strength are each non-negative quantities.

Throughout this module, symbols representing vectors will generally be printed in bold. So, for example, we may indicate a force by $\mathbf{F}$ and a velocity by $\mathbf{v}$. When writing by hand, the same indication is given by underlining the symbol, e.g. $\underline{v}$.

Underlining symbols that represent vectors

It is very important to underline handwritten symbols that denote vectors. If you fail to do so, those reading your work may not be able to tell that you are referring to a vector, and you may be penalised.

The magnitude of a vector, $\mathbf{s}$ say, is best handwritten as $|\mathbf{s}|$. Sometimes in the text, when there is no possibility of ambiguity, we will simply display it as s. The magnitude of a vector is sometimes referred to as its **modulus**.

Exercise 1

The displacement from Brighton to Oxford is (approximately) 133 km North-West. Use this, together with the displacements given in the text above, to sketch a rough map showing the relative locations of London, Brighton, Oxford and Milton Keynes. According to your map, what is the approximate distance between Milton Keynes and London? How should this distance be described in terms of the displacement vector from London to Milton Keynes?

1.2 Equating vectors

Having reviewed the definition and representation of vectors, we can now begin to develop the *algebra* of vectors. This will occupy several subsections and will lead us into detailed considerations of the addition and multiplication of vectors. We begin, however, with the fundamental idea of what it means to say that two vectors are *equal*.

Recalling that a vector quantity is completely specified by its direction and magnitude, we have the following definition.

Two vectors are **equal** if they have the same direction and the same magnitude.

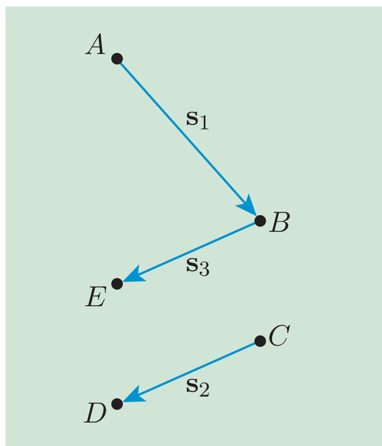


Figure 2 The displacement $\mathbf{s}_2$ is equal to the displacement $\mathbf{s}_3$

Figure 2 is similar to Figure 1 apart from an extra point E and a new displacement vector, $\mathbf{s}_3$, that stretches from B to E. The direction and distance from B to E is the same as that from C to D, so the displacement $\mathbf{s}_3$ is equal to the displacement $\mathbf{s}_2$, and we can write $\mathbf{s}_2 = \mathbf{s}_3$.

Note that the different starting points of $\mathbf{s}_2$ and $\mathbf{s}_3$ in Figure 2 do not prevent the displacements from being equal. We may use points such as B and E when we specify a displacement, but the relevant displacement, $\mathbf{s}_3$ in this case, is completely specified by its direction and magnitude; it is not in any sense 'tied' to the particular points B and E.

More generally, when we use arrows to represent any kind of vector quantity, the point at which we choose to draw the tail of the arrow is of no intrinsic significance. (Of course, the *effect* that a vector has when applied at one point may be quite different from the effect of applying the same vector at a different point, but that is irrelevant to the specification of the vector.) Thus, when considering vectors in their own right (rather than their effects), we are free to move the arrows that represent the vectors from one place to another, provided that we do not change their direction or magnitude. This is a principle that we will use in the next section, when we consider the scaling and addition of vectors.

1.3 Scaling and adding vectors

Scaling vectors

The two arrows in Figure 3(a) represent vectors $\mathbf{g}$ and $\mathbf{h}$. Both vectors point in the same direction and are therefore said to be **parallel**. The arrow representing $\mathbf{h}$ is twice as long as the arrow representing $\mathbf{g}$, so we write $\mathbf{h} = 2\mathbf{g}$ and say that the vector $\mathbf{h}$ is equal to $\mathbf{g}$ *scaled* by 2. More generally, if $\mathbf{v}$ is a vector and m is a positive scalar, then the scaled vector $\mathbf{p} = m\mathbf{v}$ is a vector in the same direction as $\mathbf{v}$ but with magnitude $m|\mathbf{v}|$.

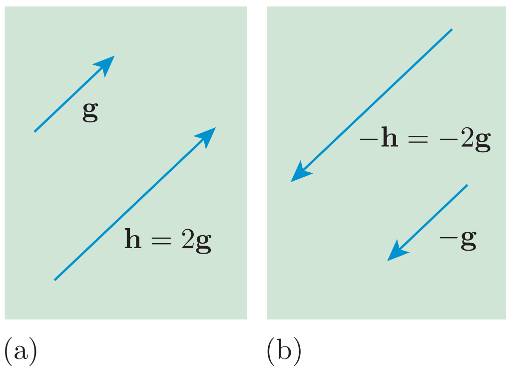


Figure 3 Multiplying a vector by a scalar

We can also scale a vector by a negative value. If m is a negative scalar, the vector $m\mathbf{v}$ will still have a positive magnitude $|m|\,|\mathbf{v}|$ (magnitudes are never negative), but $m\mathbf{v}$ will point in the *opposite* direction to $\mathbf{v}$. Vectors that point in opposite directions are often said to be **antiparallel**.

A special case of scaling occurs when $m = -1$. The negatively scaled vector $(-1)\mathbf{v}$ is normally written as $-\mathbf{v}$. The two arrows in Figure 3(b) therefore represent the scaled vectors $-\mathbf{g}$ and $-\mathbf{h} = -2\mathbf{g}$.

What happens when we scale a vector by zero (i.e. when $m = 0$)? The above definitions imply that the result is a vector of zero magnitude: this is called the **zero vector**, and is represented by the bold symbol $\mathbf{0}$.

We do not usually associate a direction with the zero vector, as all zero vectors are equal.

Collecting together the results of this subsection, we have the following.

Scaling a vector

For any vector $\mathbf{v}$ and any scalar m, the result of **scaling** $\mathbf{v}$ by m is represented by the product $m\mathbf{v}$ and is the vector with magnitude $|m|\,|\mathbf{v}|$ that is:

- parallel to $\mathbf{v}$ if $m > 0$
- antiparallel to $\mathbf{v}$ if $m < 0$
- the zero vector $\mathbf{0}$ if $m = 0$.

Adding vectors

Consider two successive displacements indicated in Figure 4. The first, $\mathbf{s}_1$, takes us from P to Q. The second, $\mathbf{s}_2$, takes us from Q to R. The net result is described by the single displacement $\mathbf{s}$, which takes us directly from P to R. In this sort of situation we interpret $\mathbf{s}$ as the result of *adding* the displacements $\mathbf{s}_1$ and $\mathbf{s}_2$, so we write

$$\mathbf{s} = \mathbf{s}_1 + \mathbf{s}_2 \quad \text{or} \quad \overrightarrow{PR} = \overrightarrow{PQ} + \overrightarrow{QR}.$$

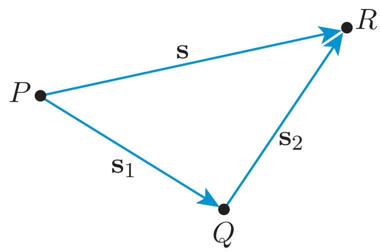


Figure 4 Two successive displacements $\mathbf{s}_1$ and $\mathbf{s}_2$, and their net result, $\mathbf{s}$

Note that in adding the displacements we have had to take the directions of $\mathbf{s}_1$ and $\mathbf{s}_2$ into account; we were not able simply to add their magnitudes. To emphasise this we refer to the general process of adding vectors as **vector addition**, and we call the outcome the **resultant**.

In the simple case of vector addition considered above, we could determine the resultant by using a triangular diagram. This graphical approach provides the basis of a more general *triangle rule* that we can use to add any two vectors of the same physical type – two displacements say, or two velocities. The triangle rule takes into account our freedom to locate the tail of a vector arrow wherever we want; it can be stated as follows.

The triangle rule

To add a vector $\mathbf{a}$ to a vector $\mathbf{b}$ of the same physical type, first draw an arrow to represent $\mathbf{a}$, then draw an arrow to represent $\mathbf{b}$ so that its tail is coincident with the head of the arrow representing $\mathbf{a}$. An arrow drawn from the tail of $\mathbf{a}$ to the head of $\mathbf{b}$ then represents the *resultant* of the *vector addition* $\mathbf{a} + \mathbf{b}$ (see Figure 5).

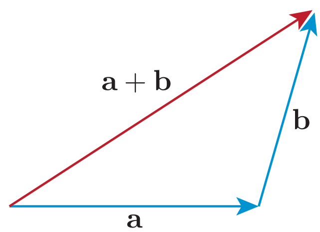


Figure 5 The triangle rule for vector addition

Combining this interpretation of vector addition with what has already been said about scaling a vector allows us to make sense of **vector subtraction**, since we can write

$$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-1)\mathbf{b}.$$

As you can see, the expression on the right is just the vector sum of $\mathbf{a}$ and the negatively scaled vector $-\mathbf{b}$ (which is antiparallel to $\mathbf{b}$). A trivial case occurs when $\mathbf{a} = \mathbf{b}$, which gives the very natural-looking vector equation

$$\mathbf{a} - \mathbf{a} = \mathbf{0}.$$

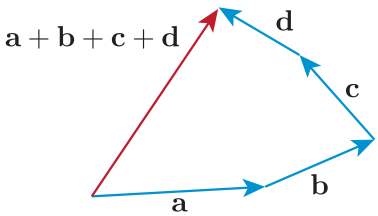
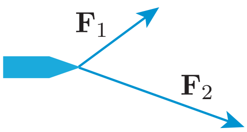


Figure 6 Extending the triangle rule to more than two vectors

The definition of the sum of two vectors is readily extended to give the sum of many vectors, $\mathbf{a}+\mathbf{b}+\mathbf{c}+\cdots$. We simply use the triangle rule to find the sum $\mathbf{s}=\mathbf{a}+\mathbf{b}$, and then use it again to add $\mathbf{c}$ to $\mathbf{s}$, and we keep on going in this way until all the vectors in the sum have been added. Figure 6 illustrates the process. In effect, the arrows are strung together in a chain, head to tail, with the head of one arrow coincident with the tail of the next. The arrow representing the final resultant vector is then obtained by joining the tail of the first arrow to the head of the last.

It is clear from the triangle rule that $\mathbf{a}+\mathbf{b}=\mathbf{b}+\mathbf{a}$, and it follows that when adding (and subtracting) several vectors, the order in which we carry out the various additions and subtractions has no influence on the final result. We describe this by saying that vector addition is **commutative**.



is equivalent to

provided that

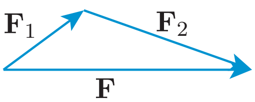


Figure 7 Forces $\mathbf{F}_1$ and $\mathbf{F}_2$ are equivalent to the resultant $\mathbf{F}$

Vector addition: a real-world perspective

The ability to add vectors together using the triangle rule is of great practical use. For example, suppose that two different forces, $\mathbf{F}_1$ and $\mathbf{F}_2$, act at a single point on an object, as shown in Figure 7. The net effect of those forces is the same as that of the single force given by the vector sum

$$\mathbf{F}=\mathbf{F}_1+\mathbf{F}_2,$$

which can be determined using the triangle rule.

Velocities may be added in the same way. Suppose that an aeroplane is travelling with velocity $\mathbf{u}$ relative to the surrounding air. Further suppose that the air is moving with velocity $\mathbf{w}$ relative to the ground. The velocity of the aeroplane relative to the ground will then be given by the vector sum

$$\mathbf{v}=\mathbf{u}+\mathbf{w},$$

which can be determined using the triangle rule.

Example 1

The vector $\mathbf{a}$ points to the East and has magnitude 3. The vector $\mathbf{b}$ points to the North and has magnitude 2.

(a) Draw a diagram (with North at the top and East to the right) showing arrows representing $\mathbf{a}$, $2\mathbf{b}$ and $\mathbf{c}=\mathbf{a}+2\mathbf{b}$.

(b) What is the magnitude of the vector $\mathbf{c}$?

(c) What is the angle between the direction of $\mathbf{c}$ and the direction of $\mathbf{a}$? Specify your answer as an angle between 0 and $\pi/2$ radians.

Solution

(a) An appropriate diagram is shown in Figure 8.

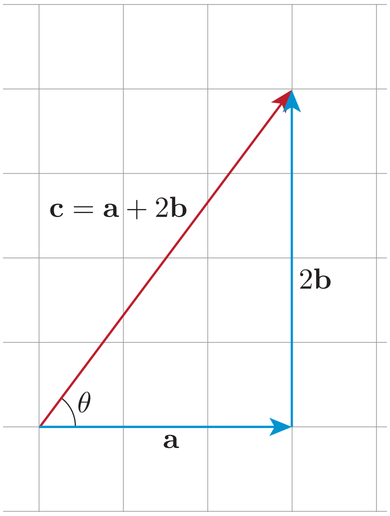


Figure 8

The arrow for $\mathbf{a}$ is 3 units long. The arrow for $\mathbf{b}$ is 2 units long, so the arrow for $2\mathbf{b}$ is 4 units long. The arrows for $\mathbf{a}$ and $2\mathbf{b}$ are mutually perpendicular. The arrow for $\mathbf{c}$ is then as shown in the diagram.

(b) The arrow for $\mathbf{c}$ forms the hypotenuse of a right-angled triangle. Using Pythagoras's theorem, the magnitude of $\mathbf{c}$ is

$$|\mathbf{c}| = \sqrt{3^2 + 4^2} = 5.$$

(c) Let θ be the angle between the directions of $\mathbf{c}$ and $\mathbf{a}$. Then the diagram shows that $\tan\theta = 4/3$, so $\theta = \arctan(4/3) = 0.927$ radians.

Exercise 2

(a) Find the magnitude of the vector $\mathbf{h} = 2\mathbf{a} - 3\mathbf{b}$, where $\mathbf{a}$ and $\mathbf{b}$ are as defined in Example 1.

(b) What is the angle between the direction of $\mathbf{h}$ and the direction of $2\mathbf{a}$? Specify your answer as an angle between 0 and $\pi/2$ radians.

1.4 Cartesian components and basic vector algebra

Despite the usefulness of the triangle rule, graphical methods are not generally very accurate and are difficult to apply in three dimensions. For these reasons, this subsection will introduce methods based on the use of *Cartesian components of vectors* that will enable us to define the equality, scaling and addition of vectors in algebraic terms. We start with a review of Cartesian coordinates.

Cartesian coordinate systems

Cartesian coordinates are named in honour of philosopher and mathematician René Descartes (1596–1650), who pioneered the geometric use of coordinates in his 1637 book *La Géométrie*.

Figure 9(a) shows a three-dimensional system of **Cartesian coordinates**. Such a system consists of three mutually perpendicular **axes** that meet at a point O called the **origin**. The axes are conventionally labelled x, y and z, with the z-axis oriented vertically. Each axis is given a positive sense, as indicated by the single arrowhead drawn on that axis. This gives a unique meaning to terms such as *positive x-direction* (i.e. parallel to the arrowed direction on the x-axis), and *negative z-direction* (i.e. antiparallel to the arrowed direction on the z-axis).

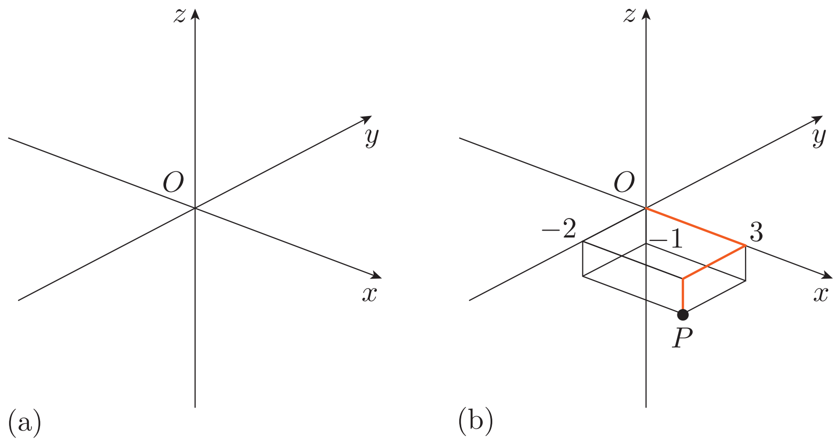


Figure 9 (a) A three-dimensional Cartesian coordinate system with origin at the point O. (b) The coordinates of any point P may be read from the axes, and described by an ordered triple (x, y, z).

If number lines are drawn along each axis, with 0 at the origin and the numbers increasing in the positive direction, it becomes possible to associate a numerical value called a **coordinate** with every point on each axis. As indicated in Figure 9(b), any three values of the x-, y- and z-coordinates, such as $x = 3$, $y = -2$ and $z = -1$, will then determine a unique point in space, such as P. Moreover, every point in three-dimensional space will correspond to a unique set of the three coordinate values. (That's what we mean by saying that space has *three* dimensions.) We can now agree to indicate any specified point by presenting its three coordinate values $x = x_1$, $y = y_1$ and $z = z_1$ as an **ordered triple** of values (x_1, y_1, z_1), always giving the three coordinates in the same conventional order, separated by commas and enclosed in round brackets. Using that convention, we can represent the point P by $(3, -2, -1)$ and say that the origin O is at the point $(0, 0, 0)$. A similar system may be applied to two dimensions by simply omitting the third coordinate and representing a point by an *ordered pair* of values such as (x_1, y_1).

In many physical situations it is necessary to associate coordinates with measured distances in specified directions. This can be done by multiplying each coordinate by an appropriate unit of measurement, such as the metre (m).

Right-handed systems of coordinates

When working in three dimensions there are two fundamentally different ways of arranging three mutually perpendicular axes. The resulting systems of Cartesian coordinates are described as *right-handed systems* and *left-handed systems*; it is important to know how to tell them apart since, by convention, we use only right-handed systems unless there is a very good reason to do otherwise. Figure 10 shows a **right-handed system** of coordinates, and its caption describes the *right-hand rule* that can be used to distinguish such a system from its left-handed counterpart.

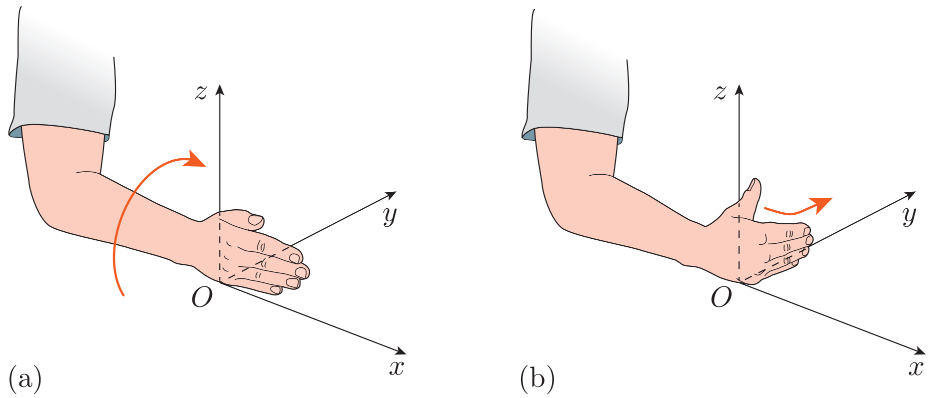


Figure 10 The handedness of a given Cartesian system can be checked using the **right-hand rule**. (a) Point the straightened fingers of your right hand in the direction of the positive x-axis, and rotate your wrist until you find that you can bend your fingers in the direction of the positive y-axis. (b) Extend the thumb of your right hand. If it points in the direction of the positive z-axis, the frame is right-handed. (If your thumb points in the negative z-direction, the system is left-handed.)

Using right-handed systems of Cartesian coordinates

When using Cartesian coordinates in three dimensions, it is conventional to work with right-handed systems.

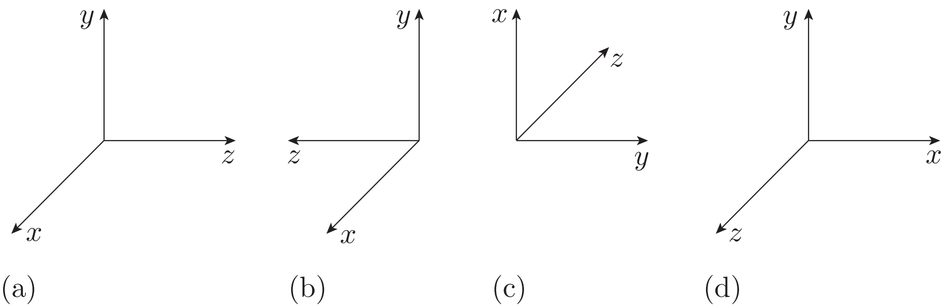
Conventionally oriented two-dimensional axes are also described as right-handed.

Exercise 3

Imagine that you are standing with your feet at the origin of a three-dimensional system of Cartesian coordinates. Your head is at some point on the positive z-axis, and you are looking into the region between the positive x-axis and the positive y-axis. Suppose that the system is right-handed and the positive x-axis is on your right. What will be on your left – the positive y-axis or the negative y-axis?

Exercise 4

Which of the sets of perpendicular axes in the figure below define right-handed coordinate systems?



The x-axis points out of the plane of the page in (a) and (b). The z-axis points respectively into and out of the plane of the page in (c) and (d).

Cartesian unit vectors

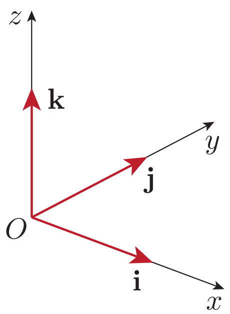


Figure 11 The unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ each have magnitude 1 and respectively point in the positive x-, y- and z-directions

When working with vectors in a region described by a right-handed system of Cartesian coordinates, it is often helpful to introduce a set of three special vectors, one directed parallel to each positive coordinate axis, and each having magnitude 1. These three vectors are called **Cartesian unit vectors** and are illustrated in Figure 11; they are conventionally labelled $\mathbf{i}$ in the positive x-direction, $\mathbf{j}$ in the positive y-direction, and $\mathbf{k}$ in the positive z-direction. For neatness, $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ have been drawn along the axes, with their tails at the origin but, as you know, that is not essential: the tails can be placed anywhere because the vectors are completely specified by their direction and magnitude. Note that the magnitude of a unit vector really is the numerical quantity 1; there are no units.

The great merit of unit vectors is that they can be easily scaled to produce vectors of any desired magnitude in any of the positive or negative coordinate directions. For example, $6\mathbf{i}$ is a vector in the positive x-direction with magnitude $|6|\,|\mathbf{i}| = 6 \times 1 = 6$. Similarly, $-3\mathbf{k}$ is a vector in the negative z-direction with magnitude $|-3|\,|\mathbf{k}| = 3 \times 1 = 3$. More generally, if λ is a non-zero scalar, the direction of $\lambda\mathbf{j}$ will depend on whether λ is greater than zero or less than zero; in either case, the magnitude of $\lambda\mathbf{j}$ will be just $|\lambda|$.

Even in those cases where we associate coordinates with physical lengths by multiplying them by units such as the metre, it is still the case that unit vectors have magnitude 1, *not* 1 metre. So, for example, for the displacement vector $\mathbf{s} = (2.5\,\mathrm{m})\,\mathbf{i}$, we say that its magnitude is $|\mathbf{s}| = |(2.5\,\mathrm{m})\,\mathbf{i}| = |2.5\,\mathrm{m}|\,|\mathbf{i}| = 2.5\,\mathrm{m} \times 1 = 2.5\,\mathrm{m}$.

Cartesian components

Now, the crucial observation is that in three dimensions any vector can be represented by a *linear combination* of the unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$. This is indicated in Figure 12 for a vector $\mathbf{a}$, which is depicted as the vector sum of the three mutually perpendicular vectors $a_x\mathbf{i}$, $a_y\mathbf{j}$ and $a_z\mathbf{k}$. Algebraically, we can write this sum as

$$\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}.$$

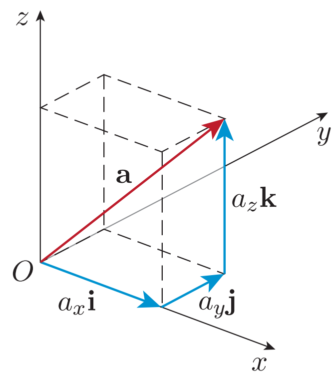


Figure 12 The vector $\mathbf{a}$ is the vector sum of the three mutually perpendicular component vectors $a_x\mathbf{i}$, $a_y\mathbf{j}$ and $a_z\mathbf{k}$

This is called the **Cartesian component form** of $\mathbf{a}$. The three vectors $a_x\mathbf{i}$, $a_y\mathbf{j}$ and $a_z\mathbf{k}$ are called the **Cartesian component vectors** of $\mathbf{a}$. However, each of those component vectors is itself the result of multiplying a Cartesian unit vector by a scalar. The three scalars involved, a_x, a_y and a_z, are called the **Cartesian scalar components** or simply **Cartesian components** of $\mathbf{a}$. In Figure 12 each Cartesian scalar component is positive, but in the general case each may be positive, zero or negative. Scalar components are referred to more frequently than component vectors, so any general references to 'components' or even 'Cartesian components' should always be interpreted as 'Cartesian scalar components' unless there is a clear indication to the contrary. With this in mind we will usually refer to a_x as the x-component of $\mathbf{a}$, a_y as the y-component, and a_z as the z-component.

In three dimensions, a vector can often be most conveniently specified in terms of its components. For a vector $\mathbf{a}$ this might be done using the linear combination $a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$ or an ordered triple (a_x, a_y, a_z). In either case, the vector is said to be in **component form**. We thus have two equivalent ways of writing a vector in component form.

Note that the ordered triple notation for a vector is identical to that for the coordinates of a point.

$$\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}, \tag{1}$$

or equivalently,

$$\mathbf{a} = (a_x, a_y, a_z). \tag{2}$$

If a vector $\mathbf{a}$ has a known direction and a known magnitude a, then we can use trigonometry to determine its scalar components. The general procedure is illustrated in Figure 13 for the case of the x-component. It shows that

$$a_x = a\cos\theta_x, \quad \text{where } 0 \le \theta_x \le \pi, \tag{3}$$

where θ_x is the angle between the direction of $\mathbf{a}$ and the positive x-direction. Similar formulas, with the analogous angles θ_y and θ_z, give the y- and z-components a_y and a_z.

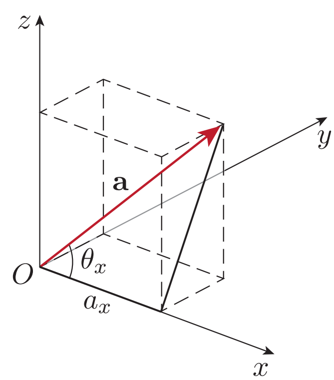


Figure 13 Finding the x-component of a vector

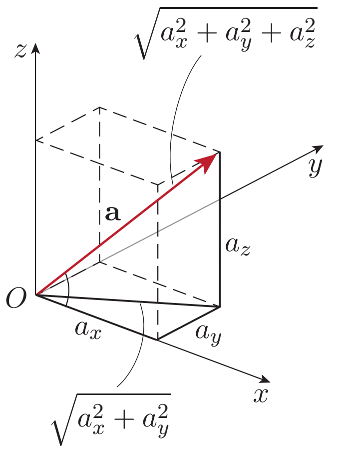


Figure 14 Finding the magnitude of a vector with known components

Conversely, if we know the component form of a vector $\mathbf{a}$, so the values of a_x, a_y and a_z are known, then it is easy to determine the magnitude and direction of $\mathbf{a}$. Using Pythagoras's theorem twice, as in Figure 14, shows that the magnitude of $\mathbf{a}$ is given by

$$a = |\mathbf{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2}. \tag{4}$$

Substituting this result into equation (3) and rearranging shows that the angle θ_x between $\mathbf{a}$ and the x-axis is given by

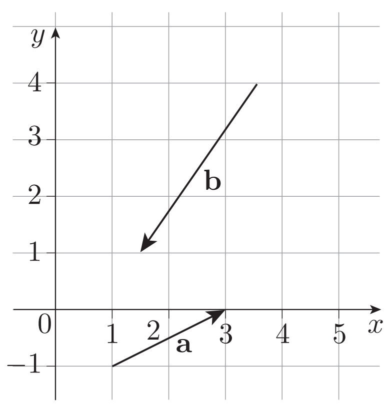
$$\cos\theta_x = \frac{a_x}{a} = \frac{a_x}{\sqrt{a_x^2 + a_y^2 + a_z^2}}, \quad \text{where } 0 \leq \theta_x \leq \pi. \tag{5}$$

Again, similar results will apply in the y- and z-directions, giving analogous expressions for $\cos\theta_y$ and $\cos\theta_z$. These three cosines will uniquely determine the direction of any non-zero vector.

Incidentally, you may have noticed that in Figures 12, 13 and 14, the vector $\mathbf{a}$ has been drawn with its tail at the origin, but this, of course, is not essential. The results that we have quoted depend only on the components of vectors and not on the point that we have chosen to represent the origin of our coordinate system.

We should also note that a common way of specifying a point with coordinates (x, y, z) is in terms of a displacement from the origin of a Cartesian coordinate system to the point. Such a vector is referred to as the **position vector** of the given point and is usually represented by the symbol $\mathbf{r}$. The components of the position vector are $r_x = x$, $r_y = y$ and $r_z = z$, so we may write $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ or equivalently $\mathbf{r} = (x, y, z)$. As far as addition and scaling are concerned, position vectors may be treated in the same way as any other displacements.

Exercise 5



The figure in the margin shows two vectors, $\mathbf{a}$ and $\mathbf{b}$, in a two-dimensional Cartesian coordinate system. In this system the components of $\mathbf{a}$ and $\mathbf{b}$ happen to be integers.

(a) Determine the components of $\mathbf{a}$ and $\mathbf{b}$ by visual inspection, then express each vector as a linear combination of unit vectors and as an ordered pair of scalar components.

(b) Use the components of $\mathbf{a}$ and $\mathbf{b}$ to determine their magnitudes, and for each vector find the angle between the direction of the vector and the positive x-direction.

Unit vectors in other directions

Given a vector **a**, it is often necessary to construct a **unit vector** in the same direction, as indicated in Figure 15. Such a general unit vector is usually denoted by $\widehat{\mathbf{a}}$; it will have magnitude 1, and is obtained by dividing the non-zero vector **a** by its own magnitude. Thus

$$\widehat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|}. \tag{6}$$

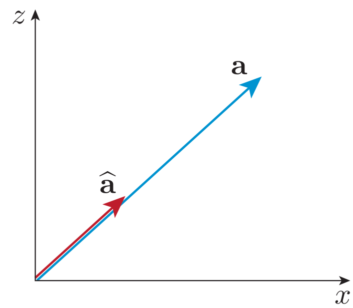


Figure 15 A unit vector $\widehat{\mathbf{a}}$ in the direction of vector **a**

The vector $\widehat{\mathbf{a}}$ is just **a** scaled by $1/|\mathbf{a}|$, so $\widehat{\mathbf{a}}$ lies in the same direction as **a** and has magnitude

$$|\widehat{\mathbf{a}}| = \left|\frac{1}{|\mathbf{a}|}\,\mathbf{a}\right| = \frac{1}{|\mathbf{a}|}\,|\mathbf{a}| = 1.$$

The x-component of the unit vector $\widehat{\mathbf{a}}$ is given by $\cos\theta_x$, the quantity described by equation (5). In fact, using its component form, we can write the unit vector in the direction of **a** as

$$\widehat{\mathbf{a}} = (\widehat{a}_x, \widehat{a}_y, \widehat{a}_z) = (\cos\theta_x, \cos\theta_y, \cos\theta_z) = \left(\frac{a_x}{a}, \frac{a_y}{a}, \frac{a_z}{a}\right). \tag{7}$$

We see that the information contained in $\widehat{\mathbf{a}}$ is just the *direction* of **a**.

As a simple illustration of this, suppose that $\mathbf{a} = (1, 2, 0)$. It then follows from equation (4) that $a = \sqrt{1^2 + 2^2} = \sqrt{5}$, so the unit vector in the direction of **a** is $\widehat{\mathbf{a}} = (1/\sqrt{5}, 2/\sqrt{5}, 0)$.

Basic vector algebra with Cartesian components

Vectors were introduced earlier as essentially geometric entities, and actions such as equating, scaling and adding vectors were all introduced in geometric terms. Now, however, following the introduction of components, we can give each of these actions an algebraic interpretation in terms of components. So, for example, we already know that two vectors $\mathbf{a} = (a_x, a_y, a_z)$ and $\mathbf{b} = (b_x, b_y, b_z)$ will be equal if they have the same direction and magnitude, but we also know that the direction and magnitude of a vector are determined by the vector's components. Consequently, the two vectors will be equal if their corresponding components are equal, so a necessary and sufficient condition for $\mathbf{a} = \mathbf{b}$ is that

$$a_x = b_x, \quad a_y = b_y, \quad a_z = b_z.$$

Similarly, the scaling of the vector **a** by the scalar λ to produce the vector $\lambda\mathbf{a}$ can be interpreted as the multiplication of each component of **a** by λ. So the operation of scaling is represented algebraically by the relation

$$\lambda\mathbf{a} = \lambda(a_x, a_y, a_z) = (\lambda a_x, \lambda a_y, \lambda a_z).$$

In a similar way, the vector sum $\mathbf{a} + \mathbf{b}$ that was introduced geometrically using the triangle rule can be reinterpreted in terms of the sum of corresponding components. So given $\mathbf{a} = (a_x, a_y, a_z)$ and $\mathbf{b} = (b_x, b_y, b_z)$, we can say that their vector sum is given by

$$\mathbf{a} + \mathbf{b} = (a_x + b_x, a_y + b_y, a_z + b_z).$$

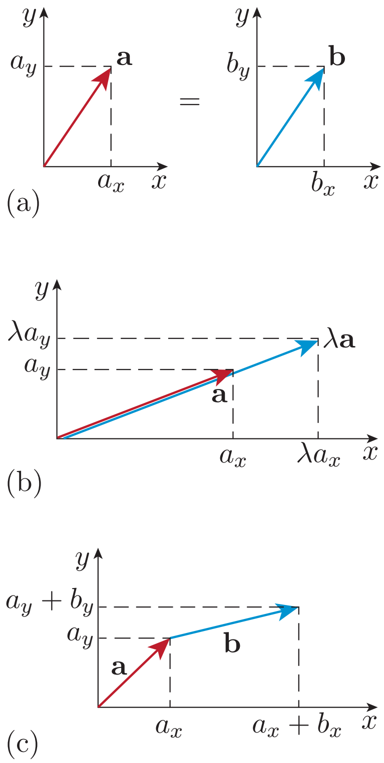


Figure 16 A component-based reinterpretation (in two dimensions) of the basic operations of vector algebra: (a) the equivalence of two vectors; (b) the scaling of a vector by a scalar; (c) the addition of two vectors

These algebraic reinterpretations are indicated graphically (in two dimensions) in Figure 16, but the transition from geometry to algebra that they constitute is of such significance that we also reproduce their three-dimensional versions in the following summary.

Basic vector algebra in terms of Cartesian components

- Given the component form of a vector $\mathbf{a} = (a_x, a_y, a_z)$, or equivalently, $\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$, the **magnitude** of $\mathbf{a}$ is given by
$$a = |\mathbf{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2}, \tag{8}$$
and the **direction** of $\mathbf{a}$ can be indicated by the **unit vector**
$$\widehat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|} = \left(\frac{a_x}{a}, \frac{a_y}{a}, \frac{a_z}{a}\right) = (\cos\theta_x, \cos\theta_y, \cos\theta_z). \tag{9}$$
- Given also a second vector $\mathbf{b} = (b_x, b_y, b_z)$, or equivalently, $\mathbf{b} = b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k}$, the vectors $\mathbf{a}$ and $\mathbf{b}$ are said to be **equal**, and we write $\mathbf{a} = \mathbf{b}$, when
$$a_x = b_x, \quad a_y = b_y, \quad a_z = b_z. \tag{10}$$
- The **scaling** of vector $\mathbf{a}$ by the scalar λ produces the scaled vector
$$\lambda\mathbf{a} = (\lambda a_x, \lambda a_y, \lambda a_z), \tag{11}$$
or equivalently,
$$\lambda\mathbf{a} = \lambda a_x\mathbf{i} + \lambda a_y\mathbf{j} + \lambda a_z\mathbf{k}. \tag{12}$$
- The vector **addition** of the vectors $\mathbf{a}$ and $\mathbf{b}$ produces the **resultant**
$$\mathbf{a} + \mathbf{b} = (a_x + b_x, a_y + b_y, a_z + b_z), \tag{13}$$
or equivalently,
$$\mathbf{a} + \mathbf{b} = (a_x + b_x)\mathbf{i} + (a_y + b_y)\mathbf{j} + (a_z + b_z)\mathbf{k}. \tag{14}$$

These first steps in vector algebra naturally suggest that we can go further, based on the further exploitation of Cartesian components. We will do this in the next section when we discuss two extremely useful ways of forming products of vectors.

Example 2

Let $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{b} = 2\mathbf{i} - 3\mathbf{j} - \mathbf{k}$ and $\mathbf{c} = 3\mathbf{i} + \mathbf{k}$.

(a) Express $\mathbf{d} = 2\mathbf{a} - 3\mathbf{b}$ and $\mathbf{e} = \mathbf{a} - 2\mathbf{b} + 4\mathbf{c}$ in component form.

(b) Find the magnitudes of the vectors $\mathbf{d}$ and $\mathbf{e}$.

(c) Evaluate $|\mathbf{a}|$, and write down a unit vector in the direction of $\mathbf{a}$.

(d) Find the components of a vector $\mathbf{g}$ such that $\mathbf{a} + \mathbf{g} = \mathbf{b}$.

Solution

(a) $\mathbf{d} = 2(\mathbf{i}+\mathbf{j}+\mathbf{k}) - 3(2\mathbf{i}-3\mathbf{j}-\mathbf{k}) = -4\mathbf{i}+11\mathbf{j}+5\mathbf{k}$,

$\mathbf{e} = (\mathbf{i}+\mathbf{j}+\mathbf{k}) - 2(2\mathbf{i}-3\mathbf{j}-\mathbf{k}) + 4(3\mathbf{i}+\mathbf{k}) = 9\mathbf{i}+7\mathbf{j}+7\mathbf{k}$.

(b) Using equation (8),

$$|\mathbf{d}| = \sqrt{(-4)^2+11^2+5^2} = \sqrt{162} = 9\sqrt{2},$$
$$|\mathbf{e}| = \sqrt{9^2+7^2+7^2} = \sqrt{179}.$$

(c) $|\mathbf{a}| = \sqrt{1^2+1^2+1^2} = \sqrt{3}$.

Using equation (9), a unit vector in the direction of $\mathbf{a}$ is

$$\widehat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|} = \frac{1}{\sqrt{3}}(\mathbf{i}+\mathbf{j}+\mathbf{k}).$$

(d) If $\mathbf{a}+\mathbf{g}=\mathbf{b}$, then

$$\begin{aligned}\mathbf{g} = \mathbf{b}-\mathbf{a} &= (2\mathbf{i}-3\mathbf{j}-\mathbf{k}) - (\mathbf{i}+\mathbf{j}+\mathbf{k})\\ &= \mathbf{i}-4\mathbf{j}-2\mathbf{k}.\end{aligned}$$

Thus the scalar components of $\mathbf{g}$ are $g_x = 1$, $g_y = -4$ and $g_z = -2$.

Exercise 6

Let $\mathbf{a} = 2\mathbf{i}-\mathbf{j}$, $\mathbf{b} = \mathbf{i}+3\mathbf{j}+5\mathbf{k}$ and $\mathbf{c} = \mathbf{j}-2\mathbf{k}$.

(a) Find the magnitudes of $\mathbf{a}$ and $\mathbf{b}$.

(b) Find the values of θ_x, θ_y and θ_z, giving the direction of $\mathbf{a}$.

(c) Find the vectors $\mathbf{a}+\mathbf{b}$, $2\mathbf{a}-\mathbf{b}$ and $\mathbf{c}+2\mathbf{b}-3\mathbf{a}$ in component form.

(d) For the displacement vector $\overrightarrow{PQ} = 2\mathbf{a}-\mathbf{b}$, where the point P is $(0,2,3)$, find the endpoint Q.

(e) For the displacement vector $\overrightarrow{RS} = \mathbf{a}+2\mathbf{b}$, where the point R is $(1,1,0)$, find the endpoint S.

Exercise 7

Confirm that the unit vector

$$\widehat{\mathbf{a}} = \left(\frac{a_x}{a}, \frac{a_y}{a}, \frac{a_z}{a}\right)$$

does indeed have magnitude 1.

Vector equation of a straight line

One useful application of position vectors (in two or three dimensions) is in obtaining a vector equation of a straight line.

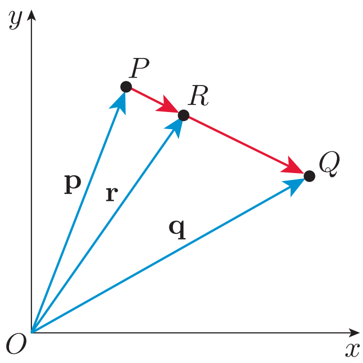


Figure 17

Example 3

Find the position vector of a point R lying on the straight-line segment PQ (see Figure 17) in terms of the position vectors of P and Q.

Solution

Using the triangle rule, the position vector $\overrightarrow{OR}$ can be written as

$$\overrightarrow{OR} = \overrightarrow{OP} + \overrightarrow{PR}.$$

Now $\overrightarrow{PR} = t\,\overrightarrow{PQ}$ for some number t, and the point R traces out the line segment PQ as t varies from 0 to 1. Thus the straight-line segment PQ is described by the vector equation

$$\overrightarrow{OR} = \overrightarrow{OP} + t\,\overrightarrow{PQ} \quad (0 \leq t \leq 1).$$

Writing $\mathbf{p} = \overrightarrow{OP}$, $\mathbf{q} = \overrightarrow{OQ}$, $\mathbf{r} = \overrightarrow{OR}$, and noting (using the triangle rule) that $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \mathbf{q} - \mathbf{p}$, this equation can also be written as

$$\mathbf{r} = \mathbf{p} + t(\mathbf{q} - \mathbf{p}) = (1 - t)\mathbf{p} + t\mathbf{q} \quad (0 \leq t \leq 1).$$

Note that if the parameter t in Example 3 is allowed to range over all the real numbers ($-\infty < t < \infty$), then the point R traces out the entire straight line of which PQ is a segment. Also note that the ideas in Example 3 are easily extended to three dimensions.

Vector equation of a straight line

If P and Q are any two distinct points on a straight line in space, with position vectors $\mathbf{p}$ and $\mathbf{q}$, respectively, then the **vector equation of the straight line** is

$$\mathbf{r}(t) = \mathbf{p} + t(\mathbf{q} - \mathbf{p}) = (1 - t)\mathbf{p} + t\mathbf{q} \quad (-\infty < t < \infty), \qquad (15)$$

where $\mathbf{r}(t)$ represents the position vector of any point on the line.

If $0 \leq t \leq 1$, then the equation represents only the line segment PQ.

Exercise 8

Write down, in component form, the vector equation of the straight line on which lie the points with Cartesian coordinates $(1, 1, 2)$ and $(2, 3, 1)$.

Vector-valued functions

Recall that a *real-valued function* $f(t)$ is an entity that gives a real value for each value of the variable t. The vector equation of a straight line introduced above, equation (15), is an example of something called a **vector-valued function**, i.e. an entity $\mathbf{r}(t)$ that gives a vector for each

value of the variable t. The components of the straight line $\mathbf{r}(t)$ in the solution to Exercise 8 were linear functions of t: $\mathbf{r}(t) = (1 + t, 1 + 2t, 2 - t)$. More generally, the components of some *curve* in space will be $\mathbf{r}(t) = (x(t), y(t), z(t))$, where $x(t)$, $y(t)$ and $z(t)$ are some general real-valued functions of t (Figure 18).

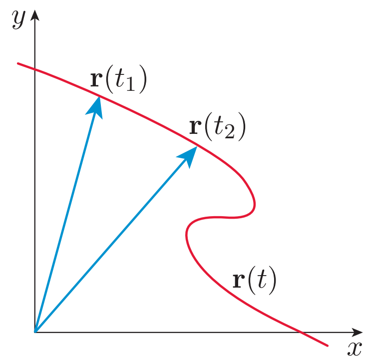


Figure 18 A curve $\mathbf{r}(t)$ in two-dimensional space. The vectors $\mathbf{r}(t_1)$ and $\mathbf{r}(t_2)$ indicate the values of $\mathbf{r}(t)$ at two values of the independent variable t.

Suppose that the position of a particle moving along some curved path is given by $\mathbf{r}(t) = (x(t), y(t), z(t))$ at time t. We want to find the velocity and acceleration of the particle, given by the derivatives of $\mathbf{r}(t)$ with respect to t. These are obtained by differentiating the components.

Example 4

A particle has position $\mathbf{r}(t) = (3t^2 - 2, t^4, -t + 1)$ at time t. Find its velocity.

Solution

The velocity of the particle is given by the derivative of $\mathbf{r}(t)$ with respect to t:

$$\mathbf{v}(t) = \dot{\mathbf{r}}(t) = \frac{d}{dt}(3t^2 - 2, t^4, -t + 1) = (6t, 4t^3, -1).$$

Exercise 9

Find the acceleration $\mathbf{a} = \dot{\mathbf{v}}$ of the particle of Example 4.

2 Products of vectors

There are two very useful ways of forming the product of two vectors $\mathbf{a}$ and $\mathbf{b}$. The first method produces a scalar quantity, represented by $\mathbf{a} \cdot \mathbf{b}$, and is called the *scalar product* or the *dot product* of the two vectors. The second method produces a vector quantity, represented by $\mathbf{a} \times \mathbf{b}$, and is called the *vector product* or the *cross product* of the two vectors.

We discuss the two products in turn, starting with the scalar product. In each case we start with a geometric view that emphasises directions and magnitudes, just as we did when defining the scaling and addition of vectors. However, we very quickly go on to express each product algebraically, in terms of components, and to examine its characteristic properties and applications.

2.1 The scalar product

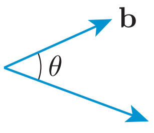


Figure 19 The angle θ $(0 \leq \theta \leq \pi)$ between two vectors

Consider two (non-zero) vectors **a** and **b**. No matter how **a** and **b** are specified (they might be given as displacements between particular points, or the velocities of particular objects), we can always use our freedom to move the arrows that represent vectors parallel to themselves to ensure that their tails meet at a point. This makes it easy to visualise the angle θ between the directions of **a** and **b**, as indicated in Figure 19, and we can always take that angle to be in the range $0 \leq \theta \leq \pi$. Using θ we can define the scalar product geometrically, as follows.

The product $\mathbf{a} \cdot \mathbf{b}$ is read as 'a dot b', and for this reason is often referred to as the **dot product**.

The **scalar product** of two vectors **a** and **b** is the scalar quantity

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos\theta, \tag{16}$$

where θ $(0 \leq \theta \leq \pi)$ is the angle between the directions of **a** and **b**.

The angle θ always lies in the range $0 \leq \theta \leq \pi$, so the value of $\mathbf{a} \cdot \mathbf{b}$ is:

- positive when θ is an acute angle (i.e. in the range $0 \leq \theta < \frac{\pi}{2}$)
- negative when θ is an obtuse angle (i.e. in the range $\frac{\pi}{2} < \theta \leq \pi$)
- zero when θ is a right angle (i.e. when $\theta = \frac{\pi}{2}$).

The last of these conditions tells us that if $\theta = \frac{\pi}{2}$, i.e. when **a** and **b** are perpendicular, then $\mathbf{a} \cdot \mathbf{b} = 0$. The definition also implies that if $\theta = 0$, i.e. when **a** and **b** are parallel, then $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| = ab$.

A special case worth remembering is that the scalar product of **a** with itself is just the square of the magnitude of **a**.

$$\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2 = a^2. \tag{17}$$

Note that the scalar product is a product in the mathematical sense, with a number of mathematically significant properties. For example, it is *commutative* so that

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}. \tag{18}$$

It also has the further properties of being *distributive over addition*, meaning that

$$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}, \tag{19}$$

This property does not obviously follow from equation (16), but will become obvious once we discuss the component form of the scalar product.

and *linear* with respect to multiplication by a scalar λ, so that

$$(\lambda\mathbf{a}) \cdot \mathbf{b} = \mathbf{a} \cdot (\lambda\mathbf{b}) = \lambda(\mathbf{a} \cdot \mathbf{b}). \tag{20}$$

These properties allow us to make sense of scalar products that involve sums and brackets.

Example 5

Expand the expression $\mathbf{x} \cdot \mathbf{y}$, given that $\mathbf{x} = 2\mathbf{u} + \mathbf{v}$ and $\mathbf{y} = \mathbf{u} - 5\mathbf{v}$. Calculate its value when $\mathbf{u}$ and $\mathbf{v}$ are perpendicular unit vectors.

Solution

Using the mathematical properties of the scalar product, we can see that

$$\begin{aligned}
\mathbf{x} \cdot \mathbf{y} &= (2\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} - 5\mathbf{v}) \\
&= (2\mathbf{u}) \cdot (\mathbf{u} - 5\mathbf{v}) + \mathbf{v} \cdot (\mathbf{u} - 5\mathbf{v}) \\
&= (2\mathbf{u}) \cdot \mathbf{u} + (2\mathbf{u}) \cdot (-5\mathbf{v}) + \mathbf{v} \cdot \mathbf{u} + \mathbf{v} \cdot (-5\mathbf{v}) \\
&= 2(\mathbf{u} \cdot \mathbf{u}) - 10(\mathbf{u} \cdot \mathbf{v}) + \mathbf{v} \cdot \mathbf{u} - 5(\mathbf{v} \cdot \mathbf{v}) \\
&= 2(\mathbf{u} \cdot \mathbf{u}) - 9(\mathbf{u} \cdot \mathbf{v}) - 5(\mathbf{v} \cdot \mathbf{v}).
\end{aligned}$$

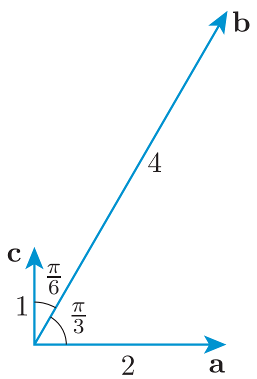
This solution is given in detail to show you there are no unexpected pitfalls when dealing with scalar products. The basic lesson is that the familiar rules of algebra still apply, so with practice you will not need to go through all these intermediate steps.

Now $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2 = 1$ and $\mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2 = 1$ when $\mathbf{u}$ and $\mathbf{v}$ are unit vectors. Furthermore, $\mathbf{u} \cdot \mathbf{v} = 0$ when $\mathbf{u}$ and $\mathbf{v}$ are perpendicular vectors. So when $\mathbf{u}$ and $\mathbf{v}$ are perpendicular unit vectors, we have

$$\mathbf{x} \cdot \mathbf{y} = 2 - 0 - 5 = -3.$$

Exercise 10

Three vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ of magnitudes 2, 4 and 1, respectively, lying in the same plane, are represented by arrows as shown in the figure below.



The angle between the vectors $\mathbf{a}$ and $\mathbf{b}$ is $\frac{\pi}{3}$ radians, and that between the vectors $\mathbf{b}$ and $\mathbf{c}$ is $\frac{\pi}{6}$ radians. Use the definition of the scalar product to find the values of $\mathbf{a} \cdot \mathbf{b}$, $\mathbf{b} \cdot \mathbf{c}$, $\mathbf{a} \cdot \mathbf{c}$ and $\mathbf{b} \cdot \mathbf{b}$.

Exercise 11

(a) Expand the expression $(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b})$.

(b) Expand the expression $|\mathbf{a} + \mathbf{b}|^2$.

Recall that $|\mathbf{a}|^2 = \mathbf{a} \cdot \mathbf{a}$.

(c) Write down the value of $\mathbf{a} \cdot \mathbf{b}$, in terms of $|\mathbf{a}|$ and $|\mathbf{b}|$, when $\mathbf{a}$ and $\mathbf{b}$ are antiparallel.

The scalar product in terms of components

Since $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are mutually perpendicular unit vectors, the following useful relations must be true.

$$\mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = \mathbf{k}\cdot\mathbf{k} = 1, \tag{21}$$

and all other scalar products of Cartesian unit vectors (such as $\mathbf{i}\cdot\mathbf{j}$) give zero.

Consequently, using the usual rules of algebra, it can be seen that

$$\begin{aligned}
\mathbf{a}\cdot\mathbf{b} &= (a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k})\cdot(b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k}) \\
&= a_x\mathbf{i}\cdot(b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k}) + a_y\mathbf{j}\cdot(b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k}) \\
&\quad + a_z\mathbf{k}\cdot(b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k}) \\
&= a_xb_x\mathbf{i}\cdot\mathbf{i} + a_xb_y\mathbf{i}\cdot\mathbf{j} + a_xb_z\mathbf{i}\cdot\mathbf{k} + a_yb_x\mathbf{j}\cdot\mathbf{i} + a_yb_y\mathbf{j}\cdot\mathbf{j} + a_yb_z\mathbf{j}\cdot\mathbf{k} \\
&\quad + a_zb_x\mathbf{k}\cdot\mathbf{i} + a_zb_y\mathbf{k}\cdot\mathbf{j} + a_zb_z\mathbf{k}\cdot\mathbf{k} \\
&= a_xb_x + a_yb_y + a_zb_z.
\end{aligned}$$

This gives us the following very useful expression for the scalar product of two vectors in terms of their components.

Component form of the scalar product

If $\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$ and $\mathbf{b} = b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k}$, then

$$\mathbf{a}\cdot\mathbf{b} = a_xb_x + a_yb_y + a_zb_z. \tag{22}$$

Many other results follow from this. For example, the relation $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{c}$ that was stated earlier becomes easy to prove. It is also easy to confirm that the Cartesian scalar components of a vector $\mathbf{a}$ are given by

$$a_x = \mathbf{i}\cdot\mathbf{a}, \quad a_y = \mathbf{j}\cdot\mathbf{a}, \quad a_z = \mathbf{k}\cdot\mathbf{a}, \tag{23}$$

and it follows from equations (16) and (22) that the angle between two non-zero vectors is given by

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|} = \frac{a_xb_x + a_yb_y + a_zb_z}{ab}, \tag{24}$$

where $0 \leq \theta \leq \pi$.

As we are now using the scalar product in a more algebraic way, this is an appropriate point at which to note that the use of algebra has allowed mathematicians to generalise the idea of what constitutes a vector and, consequently, what constitutes a scalar product of vectors. Using components, it is easy to imagine extending the definitions given earlier to more than three dimensions, but the mathematical generalisations go well beyond this. As you will see in Unit 11, even functions may be treated as 'vectors' in an appropriate space. In these generalised approaches, the

analogue of the scalar product is often called the *inner product*, and the generalisation of the condition $\mathbf{a} \cdot \mathbf{b} = 0$, the vanishing of the inner product of two (generalised) vectors, is referred to as the *orthogonality condition*. For this reason, even when dealing with 'ordinary' two- or three-dimensional vectors, you will often find that the terms **perpendicular** and **orthogonal** are used interchangeably. You will also find that the scalar product of two- or three-dimensional vectors is sometimes said to provide a **test for orthogonality**, since two non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ are orthogonal (i.e. perpendicular, in this case) if and only if $\mathbf{a} \cdot \mathbf{b} = 0$.

Example 6

Consider the vectors $\mathbf{a} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ and $\mathbf{b} = -\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$. Find the magnitudes of $\mathbf{a}$ and $\mathbf{b}$, and the angle between them.

Solution

$$|\mathbf{a}| = \sqrt{2^2 + (-3)^2 + 1^2} = \sqrt{14},$$
$$|\mathbf{b}| = \sqrt{(-1)^2 + 2^2 + 4^2} = \sqrt{21}.$$

However, from equation (22),

$$\mathbf{a} \cdot \mathbf{b} = (2 \times -1) + (-3 \times 2) + (1 \times 4) = -4,$$

so if θ is the angle between $\mathbf{a}$ and $\mathbf{b}$, then

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|} = \frac{-4}{\sqrt{14} \times \sqrt{21}} = -\frac{4}{7\sqrt{6}}.$$

The negative sign means that θ is obtuse, so $\theta \simeq 1.806$ radians.

Exercise 12

(a) If $\mathbf{a} = 4\mathbf{i} + \mathbf{j} - 5\mathbf{k}$ and $\mathbf{b} = \mathbf{i} - 3\mathbf{j} + \mathbf{k}$, show that $\mathbf{a} \cdot \mathbf{b} = -4$. What does the negative sign tell you?

(b) Are the vectors $\mathbf{c} = (3, 5, -2)$ and $\mathbf{d} = (3, -1, -2)$ orthogonal?

Exercise 13

If $\mathbf{p} = 3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$, $\mathbf{q} = -\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and $\mathbf{r} = 2\mathbf{i} - \mathbf{j} - \mathbf{k}$, and λ is a scalar, find the value of λ that makes $\mathbf{p} + \lambda\mathbf{q}$ orthogonal to $\mathbf{r}$.

Resolving a vector into perpendicular components

The process of splitting a vector into components in specified perpendicular directions is called **resolution**. So when we write $\mathbf{a}$ in component form as $\mathbf{a} = (a_x, a_y, a_z)$ or $\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$, we are showing how $\mathbf{a}$ may be *resolved* into its Cartesian components.

However, it is often useful to be able to resolve a given vector into components along perpendicular directions that are *not* aligned with the Cartesian unit vectors. This subsection will show you how to do this. First, however, we give a physical perspective on the sort of situation in which the technique is useful.

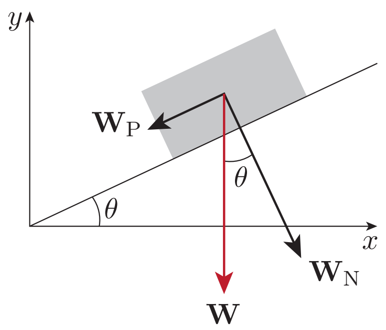


Figure 20 The vectors $\mathbf{W}_\mathrm{P}$ and $\mathbf{W}_\mathrm{N}$ are respectively parallel and normal to the plane; since they are perpendicular, $\mathbf{W} = \mathbf{W}_\mathrm{P} + \mathbf{W}_\mathrm{N}$

Resolving a vector: a physical perspective

Figure 20 shows a box on a rough wooden plane inclined at an angle θ to the horizontal. The owner of the box wants scientific advice on the maximum angle θ that can be tolerated before the box starts to slide down the plane.

The situation is described in a two-dimensional system of Cartesian coordinates with the x-axis pointing to the right and the y-axis pointing vertically upwards. In that system the weight of the box (the force exerted on the box by the Earth's gravity) points vertically downwards and is described by the vector $\mathbf{W} = -W\mathbf{j}$.

We will not go into the details of the analysis, but what is crucial is the ability to express $\mathbf{W}$ as the sum of two *orthogonal vectors*, one pointing parallel to the plane (denoted $\mathbf{W}_\mathrm{P}$), the other directed at right angles (i.e. normal) to the plane (denoted $\mathbf{W}_\mathrm{N}$). We do this by *resolving* $\mathbf{W}$ in these directions.

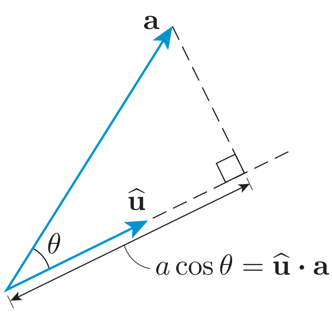


Figure 21 Finding the scalar component of $\mathbf{a}$ in the direction of a unit vector $\widehat{\mathbf{u}}$

As a general case, suppose that an arbitrary vector $\mathbf{a}$ makes an angle θ with a unit vector $\widehat{\mathbf{u}}$ (see Figure 21). Denote the scalar component of $\mathbf{a}$ in the direction of $\widehat{\mathbf{u}}$ by a_u. (Note that generally, a_u may be positive or negative, depending on the size of θ.) Simple trigonometry then shows that

$$a_u = a\cos\theta \quad (0 \le \theta \le \pi),$$

but from the definition of the scalar product, and the fact that $|\widehat{\mathbf{u}}| = 1$, we see that

$$\widehat{\mathbf{u}}\cdot\mathbf{a} = a\cos\theta \quad (0 \le \theta \le \pi).$$

This implies the following result, which is true irrespective of the sign of a_u.

The scalar component of $\mathbf{a}$ in the direction of a unit vector $\widehat{\mathbf{u}}$ is

$$a_u = \widehat{\mathbf{u}}\cdot\mathbf{a}. \tag{25}$$

Of course, this is just a generalisation of equation (23), which showed that in the Cartesian case $a_x = \mathbf{i}\cdot\mathbf{a}$, $a_y = \mathbf{j}\cdot\mathbf{a}$ and $a_z = \mathbf{k}\cdot\mathbf{a}$. Note that equation (25) can be used to find the 'components' (more generally called *projections*) of $\mathbf{a}$ in the direction of any three unit vectors $\widehat{\mathbf{u}}$, $\widehat{\mathbf{v}}$ and $\widehat{\mathbf{w}}$, but only when those unit vectors are mutually perpendicular can we say that $\mathbf{a} = a_u\widehat{\mathbf{u}} + a_v\widehat{\mathbf{v}} + a_w\widehat{\mathbf{w}}$.

Example 7

Consider Figure 22, which shows a two-dimensional vector $\mathbf{a} = (1, 3)$ and two mutually perpendicular unit vectors $\widehat{\mathbf{u}} = (1/\sqrt{2}, 1/\sqrt{2})$ and $\widehat{\mathbf{v}} = (-1/\sqrt{2}, 1/\sqrt{2})$.

(a) Show that $\mathbf{a}$ can be expressed in the form $\mathbf{a} = a_u\widehat{\mathbf{u}} + a_v\widehat{\mathbf{v}}$, with a_u and a_v given by equation (25).

(b) Calculate a_u and a_v, and hence express $\mathbf{a}$ as a linear combination of the unit vectors $\widehat{\mathbf{u}}$ and $\widehat{\mathbf{v}}$.

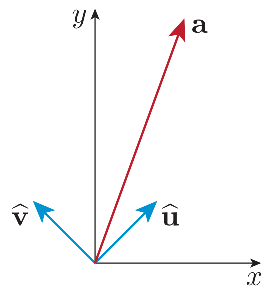


Figure 22 The vector $\mathbf{a} = (1, 3)$ and the unit vectors $\widehat{\mathbf{u}} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\widehat{\mathbf{v}} = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

Solution

(a) Since $\widehat{\mathbf{u}}$ and $\widehat{\mathbf{v}}$ are perpendicular unit vectors, we can certainly write $\mathbf{a} = \alpha\widehat{\mathbf{u}} + \beta\widehat{\mathbf{v}}$, for some values of α and β. We can determine α and β as follows.

From equation (25), the components of $\mathbf{a}$ in the $\widehat{\mathbf{u}}$ and $\widehat{\mathbf{v}}$ directions are

$$a_u = \widehat{\mathbf{u}} \cdot \mathbf{a} = \widehat{\mathbf{u}} \cdot (\alpha\,\widehat{\mathbf{u}} + \beta\,\widehat{\mathbf{v}}) = \alpha\,\widehat{\mathbf{u}} \cdot \widehat{\mathbf{u}} + \beta\,\widehat{\mathbf{u}} \cdot \widehat{\mathbf{v}} = \alpha,$$
$$a_v = \widehat{\mathbf{v}} \cdot \mathbf{a} = \widehat{\mathbf{v}} \cdot (\alpha\,\widehat{\mathbf{u}} + \beta\,\widehat{\mathbf{v}}) = \alpha\,\widehat{\mathbf{v}} \cdot \widehat{\mathbf{u}} + \beta\,\widehat{\mathbf{v}} \cdot \widehat{\mathbf{v}} = \beta.$$

Hence $\mathbf{a} = a_u\widehat{\mathbf{u}} + a_v\widehat{\mathbf{v}}$, with a_u and a_v the components of $\mathbf{a}$ in the $\widehat{\mathbf{u}}$ and $\widehat{\mathbf{v}}$ directions, respectively.

(b) We have

$$a_u = \widehat{\mathbf{u}} \cdot \mathbf{a} = \left(\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}\right) \cdot (1, 3) = \tfrac{1}{\sqrt{2}}(4) = 2\sqrt{2},$$
$$a_v = \widehat{\mathbf{v}} \cdot \mathbf{a} = \left(-\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}\right) \cdot (1, 3) = \tfrac{1}{\sqrt{2}}(2) = \sqrt{2}.$$

The vector $\mathbf{a}$ can therefore be written as the linear combination

$$\mathbf{a} = 2\sqrt{2}\,\widehat{\mathbf{u}} + \sqrt{2}\,\widehat{\mathbf{v}}.$$

Exercise 14

Consider the vectors $\mathbf{a} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ and $\mathbf{b} = -\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$.

(a) Which of the following vectors is perpendicular to $\mathbf{a}$?

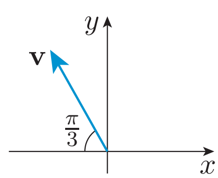
$$\mathbf{c} = -\mathbf{i} + \mathbf{j} + 3\mathbf{k}, \quad \mathbf{d} = -2\mathbf{i} + \mathbf{k}, \quad \mathbf{e} = -\mathbf{i} - \mathbf{j} - \mathbf{k}.$$

(b) Find the component of the vector $\mathbf{a} + 2\mathbf{b}$ in the direction of the displacement vector from the origin to the point $(1, 1, 1)$.

(c) Find the component of the vector $\mathbf{a} + 2\mathbf{b}$ in the direction of the vector $\mathbf{a} - 2\mathbf{b}$.

Exercise 15

A vector $\mathbf{v}$ has magnitude 4 and makes an angle of $\pi/3$ with the negative x-axis, as shown in the figure in the margin. Find the components of $\mathbf{v}$ in the $\mathbf{i}$- and $\mathbf{j}$-directions, and hence express $\mathbf{v}$ as a linear combination of $\mathbf{i}$ and $\mathbf{j}$.



Exercise 16

The three unit vectors

$$\widehat{\mathbf{u}} = \frac{1}{\sqrt{2}}(1,0,1), \quad \widehat{\mathbf{v}} = \frac{1}{\sqrt{2}}(1,0,-1) \quad \text{and} \quad \widehat{\mathbf{w}} = (0,1,0)$$

are mutually perpendicular. Express the vector $\mathbf{a} = (2,1,0)$ as a linear combination of these unit vectors.

2.2 The vector product

The *vector product* of two given vectors is a *vector*, whose direction is perpendicular to both the given vectors. It can be defined in geometric terms as follows.

The product $\mathbf{a} \times \mathbf{b}$ is read as 'a cross b', and for this reason is often referred to as the **cross product**.

The **vector product** of two vectors $\mathbf{a}$ and $\mathbf{b}$ is the vector quantity

$$\mathbf{a} \times \mathbf{b} = (|\mathbf{a}|\,|\mathbf{b}| \sin\theta)\,\widehat{\mathbf{n}}, \tag{26}$$

where θ $(0 \le \theta \le \pi)$ is the angle between the directions of $\mathbf{a}$ and $\mathbf{b}$, and $\widehat{\mathbf{n}}$ is a unit vector that is normal (i.e. perpendicular) to both $\mathbf{a}$ and $\mathbf{b}$, and whose sense is given by the right-hand rule shown in Figure 23.

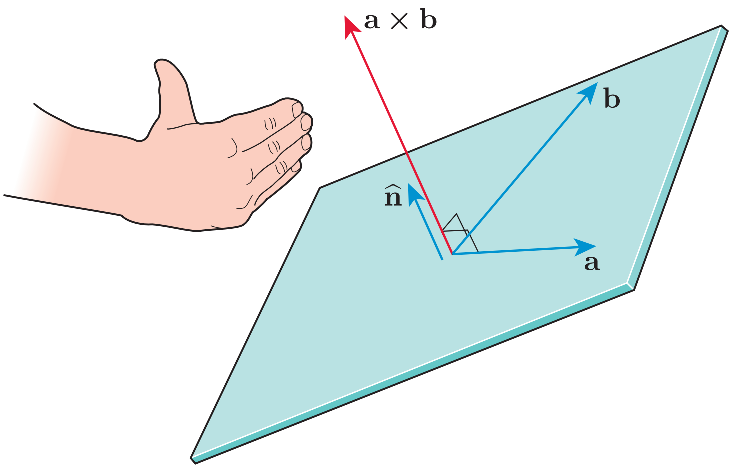


Figure 23 The **right-hand rule** for vector products. To find the sense of the unit vector $\widehat{\mathbf{n}}$ that is normal to both $\mathbf{a}$ and $\mathbf{b}$, first point the straightened fingers of your right hand in the direction of $\mathbf{a}$. Then rotate your wrist until you find that you can bend your fingers in the direction of $\mathbf{b}$. The outstretched thumb of your right hand then points in the sense of the unit vector $\widehat{\mathbf{n}}$, which has the same direction as $\mathbf{a} \times \mathbf{b}$.

Notice that $\widehat{\mathbf{n}}$ is not defined if $\mathbf{a}$ and $\mathbf{b}$ are parallel $(\theta = 0)$ or antiparallel $(\theta = \pi)$, or if $\mathbf{a}$ or $\mathbf{b}$ is the zero vector. In each of these cases, either $\sin\theta = 0$ or $|\mathbf{a}| = 0$ or $|\mathbf{b}| = 0$, so $\mathbf{a} \times \mathbf{b} = \mathbf{0}$, the zero vector.

A special case worth remembering is that the vector product of a vector $\mathbf{a}$ with itself is the zero vector: $\mathbf{a} \times \mathbf{a} = \mathbf{0}$.

More generally, since the quantity $\widehat{\mathbf{n}}$ that determines the direction of $\mathbf{a} \times \mathbf{b}$ is a unit vector, it follows that the magnitude of $\mathbf{a} \times \mathbf{b}$ is simply given by $|\mathbf{a}|\,|\mathbf{b}|\sin\theta$.

The vector product, like the scalar product, is *distributive over vector addition*, so we can say that

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c}). \tag{27}$$

This result is not easily derived from equation (26).

It is also *linear* with respect to multiplication by a scalar λ, so that

$$\mathbf{a} \times (\lambda\mathbf{b}) = \lambda\mathbf{a} \times \mathbf{b}. \tag{28}$$

These properties again allow us to expand expressions that involve sums and brackets in the usual way.

However, unlike the scalar product, the vector product is *not* commutative. This means that for vectors that are non-zero and neither parallel nor antiparallel, $\mathbf{a} \times \mathbf{b} \neq \mathbf{b} \times \mathbf{a}$. The reason for this is the use of the right-hand rule to determine the sense of $\widehat{\mathbf{n}}$. If we make $\mathbf{b}$ the first term in the product, the right-hand rule will tell us to reverse the sense of $\widehat{\mathbf{n}}$, and that means changing the sign of the vector product, so

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}. \tag{29}$$

This *non-commutativity* is a very important distinction that should always be kept in mind.

Example 8

Using the definition of the vector product, confirm that $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, $\mathbf{j} \times \mathbf{k} = \mathbf{i}$ and $\mathbf{k} \times \mathbf{i} = \mathbf{j}$.

Solution

Using the definition of the vector product (including the right-hand rule), and recalling that unit vectors have magnitude 1,

$$\mathbf{i} \times \mathbf{j} = \left(|\mathbf{i}|\,|\mathbf{j}|\sin\tfrac{\pi}{2}\right)\mathbf{k} = \mathbf{k}.$$

Similarly,

$$\mathbf{j} \times \mathbf{k} = \mathbf{i} \quad \text{and} \quad \mathbf{k} \times \mathbf{i} = \mathbf{j}.$$

Exercise 17

(a) Confirm that $\mathbf{j} \times \mathbf{i} = -\mathbf{k}$, $\mathbf{k} \times \mathbf{j} = -\mathbf{i}$ and $\mathbf{i} \times \mathbf{k} = -\mathbf{j}$.

(b) State and justify the value of $\mathbf{i} \times \mathbf{i}$, $\mathbf{j} \times \mathbf{j}$ and $\mathbf{k} \times \mathbf{k}$.

(c) Expand and simplify $(\mathbf{i} \times (\mathbf{i} + \mathbf{k})) - ((\mathbf{i} + \mathbf{j}) \times \mathbf{k})$.

(d) Expand and simplify $(\mathbf{i} + \mathbf{k}) \times (\mathbf{i} + \mathbf{j} + \mathbf{k})$.

The vector product in terms of components

The best way to get better insight into the vector product is to express it in terms of components. This will again mark an important transition from an approach that is primarily geometric to one that is more algebraic. Fundamental to this development are the results concerning the vector products of unit vectors that were discussed in the last subsection.

$$\mathbf{i}\times\mathbf{j}=\mathbf{k},\quad \mathbf{j}\times\mathbf{k}=\mathbf{i},\quad \mathbf{k}\times\mathbf{i}=\mathbf{j}, \tag{30}$$

$$\mathbf{j}\times\mathbf{i}=-\mathbf{k},\quad \mathbf{k}\times\mathbf{j}=-\mathbf{i},\quad \mathbf{i}\times\mathbf{k}=-\mathbf{j}, \tag{31}$$

and all other vector products of pairs of Cartesian unit vectors give $\mathbf{0}$.

Using these results, together with the familiar rules of algebra but taking care not to change the order of vectors in any vector product, it can be seen that

$$\begin{aligned}\mathbf{a}\times\mathbf{b} &= (a_x\mathbf{i}+a_y\mathbf{j}+a_z\mathbf{k})\times(b_x\mathbf{i}+b_y\mathbf{j}+b_z\mathbf{k})\\ &= a_xb_x\mathbf{i}\times\mathbf{i}+a_xb_y\mathbf{i}\times\mathbf{j}+a_xb_z\mathbf{i}\times\mathbf{k}+a_yb_x\mathbf{j}\times\mathbf{i}+a_yb_y\mathbf{j}\times\mathbf{j}\\ &\quad +a_yb_z\mathbf{j}\times\mathbf{k}+a_zb_x\mathbf{k}\times\mathbf{i}+a_zb_y\mathbf{k}\times\mathbf{j}+a_zb_z\mathbf{k}\times\mathbf{k}\\ &= (a_yb_z-a_zb_y)\mathbf{i}+(a_zb_x-a_xb_z)\mathbf{j}+(a_xb_y-a_yb_x)\mathbf{k}.\end{aligned}$$

This gives us two equivalent ways of expressing the vector product.

Component form of the vector product

If $\mathbf{a}=a_x\mathbf{i}+a_y\mathbf{j}+a_z\mathbf{k}$ and $\mathbf{b}=b_x\mathbf{i}+b_y\mathbf{j}+b_z\mathbf{k}$, then

$$\mathbf{a}\times\mathbf{b}=(a_yb_z-a_zb_y)\mathbf{i}+(a_zb_x-a_xb_z)\mathbf{j}+(a_xb_y-a_yb_x)\mathbf{k}, \tag{32}$$

or equivalently,

$$\mathbf{a}\times\mathbf{b}=(a_yb_z-a_zb_y,\ a_zb_x-a_xb_z,\ a_xb_y-a_yb_x). \tag{33}$$

Note that the correctness of these expressions is crucially dependent on the use of a right-handed system of coordinates.

Note the pattern in equations (32) and (33). The x-component of the product, c_x say, is given by

$$c_x=a_yb_z-a_zb_y,$$

so the first three subscripts are x, y, z in alphabetical order. In both the terms on the right above, the subscripts are y and z, but in the second term $(-a_zb_y)$ their order has reversed and the term has incurred an overall minus sign. Similar comments apply to each of the other components. In each case the first three subscripts are a *cyclic* rearrangement of x, y, z – i.e. either y, z, x or z, x, y (see Figure 24). Also, in each case the final term on the right involves a departure from cyclic reordering and incurs a minus sign as a result. Note that the x-component of $\mathbf{a}\times\mathbf{b}$ is independent of a_x and b_x, the y-component is independent of a_y and b_y, and the z-component is independent of a_z and b_z.

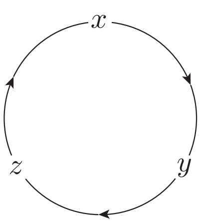


Figure 24 The cyclic basis of the vector product. The arrows indicate a positive sense; products formed in the reverse sense incur a minus sign. This provides an easy way of remembering that $\mathbf{i}\times\mathbf{j}=\mathbf{k}$ but $\mathbf{j}\times\mathbf{i}=-\mathbf{k}$, and so on.

Example 9

Evaluate $\mathbf{a} \times \mathbf{b}$, where $\mathbf{a} = (2, 3, 4)$ and $\mathbf{b} = (1, -1, -3)$.

Solution

Using equation (33), and working in ordered triple notation,

$$\begin{aligned}\mathbf{a} \times \mathbf{b} &= (2, 3, 4) \times (1, -1, -3) \\ &= (3(-3) - 4(-1), 4(1) - 2(-3), 2(-1) - 3(1)) \\ &= (-5, 10, -5).\end{aligned}$$

Since $\mathbf{a} \times \mathbf{b}$ should be perpendicular to both $\mathbf{a}$ and $\mathbf{b}$, a quick check on our calculation is provided by verifying that $(-5, 10, -5) \cdot \mathbf{a}$ and $(-5, 10, -5) \cdot \mathbf{b}$ both vanish.

Despite their symmetry, equations (32) and (33) are not easy to remember and are mainly used in formal arguments and machine calculations. When it comes to calculations performed by hand, it is usual to employ a more memorable expression based on algebraic entities called determinants that will be described in Section 4 of this unit. For that reason we mainly defer exercises that require you to evaluate vector products until Section 4, where you will be able to use the determinant method.

The vector product: a physical perspective

The vector product really opens up the world of three dimensions to physical scientists and engineers. For example, the turning effect of a force is described by a vector quantity called *torque* that is defined by a vector product. Figure 25 shows a rigid rod with one end pivoted at the origin O and the other end at the point P with position vector $\mathbf{r} = (x, y, z)$. If a force $\mathbf{F}$ is applied to the rod at the point P, its turning effect about the origin will be described by the torque $\mathbf{T} = \mathbf{r} \times \mathbf{F}$. This will be true irrespective of the relative orientations of $\mathbf{r}$ and $\mathbf{F}$.

If $\mathbf{F}$ acts at right angles to $\mathbf{r}$ (see Figure 25(a)), then the torque about the origin will be in the direction $\widehat{\mathbf{T}}$, perpendicular to $\mathbf{r}$ and $\mathbf{F}$, and will have magnitude $T = rF$. If the rod is pivoted in such a way that it cannot rotate about an axis in the direction of $\widehat{\mathbf{T}}$ but must instead rotate about some other axis through the origin, then the turning effect of $\mathbf{F}$ will be described by the *component* of $\mathbf{T}$ along the allowed axis, whatever its direction.

Similarly, if the force $\mathbf{F}$ is applied at an angle θ to the rod (see Figure 25(b)), then the magnitude of its turning effect about an axis through the origin in the direction of $\widehat{\mathbf{T}}$ will be reduced to $T = rF \sin\theta$, and will vanish completely if $\mathbf{F}$ is parallel or antiparallel to $\mathbf{r}$ since θ will then be 0 or π.

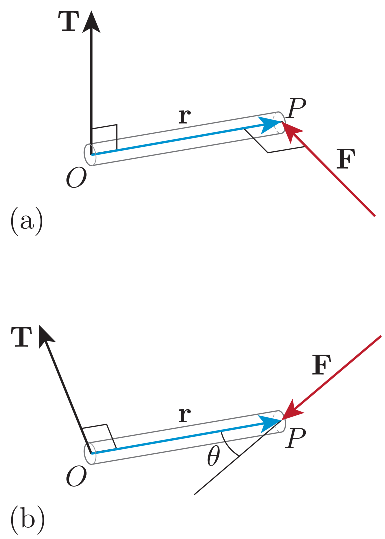


Figure 25 The torque $\mathbf{T}$ about the origin due to a force $\mathbf{F}$ applied at the point P with position vector $\mathbf{r}$: (a) when the force is at right angles to the rod; (b) when the force is at an angle θ to the rod

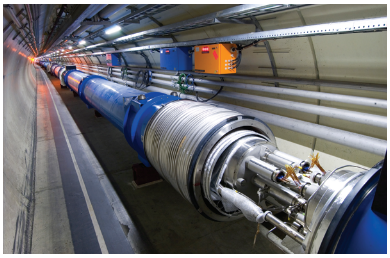


Figure 26 An open part of the LHC shows the beam tubes and bending magnets for the electrically charged particles that are accelerated by the machine

As another example, consider an electrically charged particle travelling through the powerful magnetic field inside CERN's Large Hadron Collider (LHC) (see Figure 26). Such a particle is subject to an electromagnetic force $\mathbf{F}$ that acts at right angles to both the particle's velocity vector $\mathbf{v}$ and the magnetic field vector, which is represented by $\mathbf{B}$. As the particle moves through the LHC, the direction of the electromagnetic force continuously changes. Yet no matter what the orientations of the particle's velocity and the magnetic field, the force is at all times described by the equation $\mathbf{F} = q\mathbf{v} \times \mathbf{B}$, where q is the charge on the particle.

Areas and vector products

The vector product has several useful geometric applications. The following example introduces one of them.

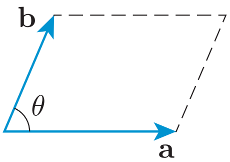


Figure 27 A parallelogram

Example 10

Any two non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ define a parallelogram, as shown in Figure 27. Find an expression for the area of the parallelogram in terms of $\mathbf{a} \times \mathbf{b}$.

Solution

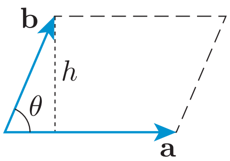


Figure 28 The perpendicular height of a parallelogram

The area A of the parallelogram defined by the two vectors $\mathbf{a}$ and $\mathbf{b}$ is equal to the product of its base length $|\mathbf{a}|$ and its perpendicular height $h = |\mathbf{b}| \sin\theta$ (see Figure 28). Thus $A = |\mathbf{a}|\,|\mathbf{b}| \sin\theta$, and this is the magnitude of $\mathbf{a} \times \mathbf{b}$. So the area of the parallelogram is

$$A = |\mathbf{a} \times \mathbf{b}|.$$

The area A of a parallelogram with sides defined by vectors $\mathbf{a}$ and $\mathbf{b}$ is given by

$$A = \text{base length} \times \text{perpendicular height} = |\mathbf{a} \times \mathbf{b}|. \qquad (34)$$

This result can be used to determine the areas of other figures, such as a *rectangle* (a special kind of parallelogram with $\mathbf{a}$ perpendicular to $\mathbf{b}$) or a *triangle*, which has half the area of the corresponding parallelogram.

Exercise 18

Using position vectors, find the area of a triangle with corners at the points $(0, 0, 0)$, $(2, 1, 1)$ and $(1, -1, -1)$.

Exercise 19

Using the vector product, confirm that the area of a parallelogram with corners at the points $(0, 0, 0)$, $(a, b, 0)$, $(c, d, 0)$ and $(a + c, b + d, 0)$ is $|ad - bc|$. Check that the formula gives the right answer in the case that the parallelogram is a rectangle, with $b = c = 0$.

Volumes and triple products

A **parallelepiped** is a solid body like a distorted brick, all of whose faces are parallelograms, as shown in Figure 29.

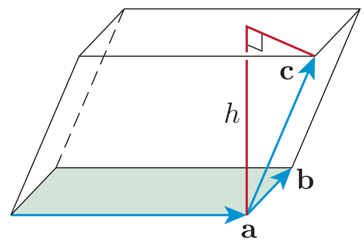


Figure 29 A parallelepiped: in the case shown, the vector **c** leans forward, out of the page, and to the right, so **c** is *not* perpendicular to either **a** or **b**

The volume V of a parallelepiped with sides defined by vectors **a**, **b** and **c** is given by

$$V = \text{base area} \times \text{vertical height} = |(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}|. \tag{35}$$

That this formula is correct can be seen as follows. The base is a parallelogram defined by the vectors **a** and **b**. The area of the base is therefore equal to the magnitude of $\mathbf{a} \times \mathbf{b}$. The vertical height h is the magnitude of the scalar component of the vector **c** in the direction perpendicular to the base. That direction is parallel or antiparallel to the direction of $\mathbf{a} \times \mathbf{b}$. So the product $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$ has a magnitude equal to the base area times the perpendicular height. We may therefore express the volume of the parallelepiped as $|(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}|$.

The expression for the volume of the parallelepiped involves the quantity $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$, which is an example of the **scalar triple product** of vectors. As the name implies, it is a scalar quantity obtained from three vectors. Such products arise in many situations and can be written in a number of equivalent ways. For example, by expressing all the products in terms of components, it is also possible to establish a **cyclic identity** according to which

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}). \tag{36}$$

Combining this with the freedom to exchange the order of vectors in the scalar product (but not in the vector product) leads to some more equivalent expressions, including the following relationship that will be used in a later unit:

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}. \tag{37}$$

Proving the equality of these two expressions involves a lot of algebraic manipulation of their components. This is straightforward but time consuming. However, the equality of their magnitudes is obvious, since either of those magnitudes can be used to describe the volume of the parallelepiped with sides defined by the three vectors **a**, **b** and **c**.

The scalar triple product (in all its equivalent forms) is not the only way of forming a meaningful product of three vectors. There is also a **vector triple product**, $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$. This produces a vector quantity that will be perpendicular to $\mathbf{c}$ and to the vector that results from $\mathbf{a} \times \mathbf{b}$.

Since it involves taking vector products, the vector triple product is naturally not commutative. Moreover, the vector triple product is not *associative* either. That is, $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ is generally different from $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$. This may seem rather surprising but is easily established as follows. Since $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ is perpendicular to the direction of $\mathbf{a} \times \mathbf{b}$, its only non-zero components must be in the plane that is perpendicular to $\mathbf{a} \times \mathbf{b}$, i.e. the plane containing $\mathbf{a}$ and $\mathbf{b}$. Similarly, the only non-zero components of $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ must be in the plane containing $\mathbf{b}$ and $\mathbf{c}$. So, provided that $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are not all in the same plane, any non-zero triple products $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ and $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ must point in different directions.

Exercise 20

Suppose that $\mathbf{a}$ and $\mathbf{b}$ have a non-zero vector product, and that $\mathbf{c}$ is a non-zero vector such that $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \mathbf{0}$. What can you say about the direction of $\mathbf{c}$?

We end our discussion of vectors with a warning. Although a powerful vector algebra has been developed with operations of scaling, addition and two kinds of multiplication, there is no **vector division**. So do not try to divide by a vector. The absence is caused by a lack of uniqueness in attempts to define vector division. When dealing with non-zero scalar quantities, the equation $ax = b$ has the unique solution $x = b/a$. When dealing with non-zero vectors, however, the corresponding equation $\mathbf{a} \cdot \mathbf{x} = b$, where b is a scalar, has many solutions. If $\mathbf{x} = \mathbf{x}_1$ is one solution (so $\mathbf{a} \cdot \mathbf{x}_1 = b$), then another solution is $\mathbf{x}_2 = \mathbf{x}_1 + \lambda\mathbf{c}$, where λ is any scalar and $\mathbf{c}$ is any vector orthogonal to $\mathbf{a}$ (so $\mathbf{a} \cdot (\lambda\mathbf{c}) = 0$, and hence $\mathbf{a} \cdot \mathbf{x}_2 = \mathbf{a} \cdot \mathbf{x}_1 = b$).

3 Matrices, vectors and linear transformations

In Sections 1 and 2 of this unit you were introduced to the algebra of vectors. Sections 3 and 4 will provide a comparable introduction to the algebra of *matrices*. As you will see, there are many deep links between vectors and matrices, but there are also some important differences. In particular, the treatment of matrices is generally more ‘algebraic’ and less ‘geometric’. This can make matrix algebra appear more abstract and harder to visualise. For that reason, rather than plunging directly into the presentation of algebraic rules for scaling, adding and multiplying

matrices, most of Section 3 will be devoted to examining a particular application of matrices that emphasises their use in geometry. This is the subject of *linear transformations of a plane*, which will be introduced below, alongside matrices themselves. This will allow us to introduce some of the basic ideas of matrix algebra in a concrete setting. The more general aspects of matrix algebra will then be introduced in Section 4.

3.1 Matrices and linear transformations

Matrices

Here are four rectangular arrays of numbers enclosed in brackets.

$$\text{(a)}\ \begin{bmatrix} 2 & -7 \\ 0 & 3 \end{bmatrix} \quad \text{(b)}\ \begin{bmatrix} -2 & 1 & 4 \\ 3 & 0 & -1 \\ 2 & 3 & 2 \end{bmatrix} \quad \text{(c)}\ \begin{bmatrix} -\pi & \pi \end{bmatrix} \quad \text{(d)}\ \begin{bmatrix} 0.8 \\ 0.3 \\ 0.6 \end{bmatrix}$$

We follow the convention of using square brackets to indicate matrices. Some texts use round brackets.

These are all examples of matrices. A matrix can be defined as follows.

A **matrix** is a rectangular array of elements (usually numbers or physical quantities) arranged in rows and columns, and enclosed in brackets. It obeys several mathematical rules that collectively comprise **matrix algebra**.

A matrix with m rows and n columns is said to be of **order** $m \times n$. We generally represent entire matrices by symbols printed in bold type, so if $\mathbf{A}$ represents an $m \times n$ matrix, we may write

$m \times n$ is read as 'm by n'; the $\times$ does *not* mean multiplication.

There is no universal convention on how to hand-write matrices. Capital letters are often used, except when the matrix represents a vector. Some people underline with a straight line, some with a curly line; some even underline twice. We leave it to you to choose a convention.

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \tag{38}$$

where a_{ij} represents the element in the ith row and the jth column. (The significance of the symbol a_{ij} is sometimes recalled using the mnemonic 'arc': element $\underline{a}$, $\underline{\text{r}}$ow i, $\underline{\text{c}}$olumn j.)

A matrix in which all of the elements are zero is called a **zero matrix** and will be denoted **0**, irrespective of its order. Matrices of order $n \times n$ have equal numbers of rows and columns and are called **square matrices**. (Examples (a) and (b) given above are square matrices.) Any matrix of order $1 \times n$ takes the form of a single row of elements and is called a **row matrix**. Similarly, a matrix of order $n \times 1$ is called a **column matrix**. (Examples (c) and (d) given above are row and column matrices, respectively.) If $n = 2$ or 3, a row matrix looks a lot like a vector presented as an ordered pair or an ordered triple, though the matrices do not contain commas. In fact, row and column matrices are often used to represent vectors algebraically. For that reason we often refer to row and column matrices as **row vectors** and **column vectors**.

Two matrices are said to be equal if they have the same order and each of the corresponding elements is equal. So, for example, $\begin{bmatrix} 3 & 8 \end{bmatrix} = \begin{bmatrix} 1+2 & 4\times 2 \end{bmatrix}$, but $\begin{bmatrix} 3 & 8 \end{bmatrix} \neq \begin{bmatrix} 3 & 8 & 0 \end{bmatrix}$ because the last matrix is not of order 1×2. More formally, we have the following.

Two matrices $\mathbf{A}$ and $\mathbf{B}$ are **equal** if they have the same order and $a_{ij} = b_{ij}$ for all $i = 1, \dots, n$ and for all $j = 1, \dots, m$.

Exercise 21

What is the order of each of the four example matrices (a), (b), (c) and (d) given at the start of this subsection?

Linear transformations

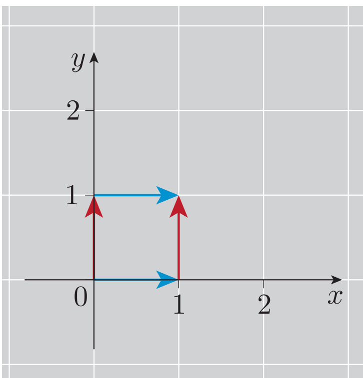


Figure 30 A plane and an overlying coordinate system. A square edged by unit vectors and a grid of straight lines have been drawn on the plane. Distortions of the plane do not affect the coordinate system.

Figure 30 represents a two-dimensional plane overlaid by a two-dimensional Cartesian coordinate system. This subsection concerns 'transformations' that affect the points in the plane but not the coordinate system. That's why the coordinates were described as 'overlaid'. You might like to think of the plane as an elastic sheet that can be rotated or stretched, while the coordinate axes are drawn on an overlying transparent plastic sheet that is not affected by the distortion of the underlying plane.

In Figure 30 we have drawn the ('overlaid') coordinate axes in black. We have also drawn a grid of straight (white) lines and a square of unit area with unit vectors along its edges. The grid lines and the vectors should be thought of as drawn on the plane (which is coloured grey), so they will all be affected by the transformation of the plane.

Using the overlying coordinates, any point in the plane can be described by a position vector (x, y). Such a point can also be represented by a row vector or by a column vector; in the case of a column vector we call it the **position column vector** and denote it by

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

The usual Cartesian unit vectors on the plane can be described by $(1, 0)$ and $(0, 1)$. These too may be represented by appropriate row or column vectors; in the case of column vectors we call them **unit column vectors**, and denote them by

$$\mathbf{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

(Note that by reusing the symbols $\mathbf{i}$ and $\mathbf{j}$ in this new way we are deliberately blurring the distinction between vectors and column matrices. It should be clear from the context which of them we are referring to, but the point is to demonstrate that matrices provide another way of dealing with vectors.)

Figure 31 shows the effect of subjecting the whole plane to a transformation that moves any point with position vector (x, y) to a new location with position vector $(ax + by, cx + dy)$, where a, b, c and d are real numbers. The particular transformation shown in Figure 31 was obtained using the values $a = 1.0$, $b = -0.9$, $c = 0.3$ and $d = 1.5$, but it is a typical example of what is generally known as a **linear transformation** of the plane. Note that although the transformation generally tends to move points in the plane, and therefore changes their coordinates, it does not change the coordinates of the point at the origin. That particular point in the plane starts with the coordinates $(x, y) = (0, 0)$, and the effect of the transformation is to make its new coordinates $(ax + by, cx + dy) = (0, 0)$, so the origin does not move at all. This is characteristic of linear transformations.

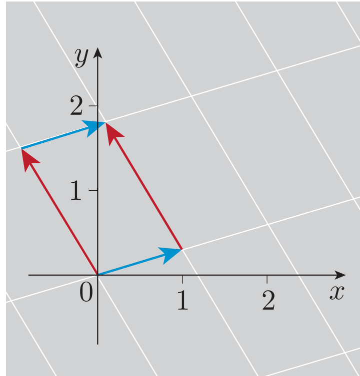


Figure 31 The effect of a linear transformation on the plane. The transformation will generally change the coordinates of points in the plane but has no effect on the coordinate system (shown in black). The origin is unaffected, and straight lines are transformed into straight lines.

Our aim now is to show how linear transformations of the plane can be represented in a very natural way using matrices. As a first step towards this goal, we introduce the following *multiplication rule* for determining the product of a 2×2 square matrix and a 2×1 column matrix. You will see later that this is a special case of the general rule for matrix multiplication.

The product of a 2×2 matrix $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and a 2×1 matrix $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ is a 2×1 matrix given by

$$\mathbf{A}\mathbf{x} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}. \tag{39}$$

At first sight this rule may seem rather arbitrary, but there is actually a very sensible pattern behind it. As indicated by the hand symbols in Figure 32(a), the expression $ax + by$ in the first row and first (and only) column of the product, is the result of adding the products of the corresponding elements in the first row of $\mathbf{A}$ and the first (and only) column of $\mathbf{x}$. Similarly, as indicated in Figure 32(b), the expression $cx + dy$ in the second row and first (and only) column of the product, is the result of adding the products of the corresponding elements in the second row of $\mathbf{A}$ and the first (and only) column of $\mathbf{x}$.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

(a)

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

(b)

Figure 32 (a) Obtaining $ax + by$; (b) obtaining $cx + dy$

If we now substitute into the matrix $\mathbf{A}$ the values $a = 1.0$, $b = -0.9$, $c = 0.3$ and $d = 1.5$ that characterise the transformation shown in Figure 31, we see that

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1.0 & -0.9 \\ 0.3 & 1.5 \end{bmatrix}.$$

The multiplication rule given in equation (39) then tells us that the product of $\mathbf{A}$ and the unit column vector $\mathbf{i}$ is

$$\mathbf{Ai} = \begin{bmatrix} 1.0 & -0.9 \\ 0.3 & 1.5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1.0 \\ 0.3 \end{bmatrix},$$

Note how naturally we can move between the vector notation $(.,.)$ and the column vector notation $\begin{bmatrix} . \\ . \end{bmatrix}$.

and this exactly describes what happens to the unit vector $(1, 0)$ when it is affected by the linear transformation of Figure 31; it becomes the vector $(1.0, 0.3)$.

Similarly, according to the multiplication rule, the product of $\mathbf{A}$ and the unit column vector $\mathbf{j}$ is

$$\mathbf{Aj} = \begin{bmatrix} 1.0 & -0.9 \\ 0.3 & 1.5 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.9 \\ 1.5 \end{bmatrix},$$

which exactly describes what happens to the unit vector $(0, 1)$ when it is affected by the linear transformation of Figure 31; it becomes the vector $(-0.9, 1.5)$.

In fact, the multiplication rule tells us that the product of $\mathbf{A}$ and a position column matrix $\mathbf{x}$ (representing a general point in the plane) is just

Note that the quantity on the extreme right is a 2×1 column vector, *not* a 2×2 matrix.

$$\mathbf{Ax} = \begin{bmatrix} 1.0 & -0.9 \\ 0.3 & 1.5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1.0x - 0.9y \\ 0.3x + 1.5y \end{bmatrix},$$

which exactly represents the general effect of the linear transformation on a position vector (x, y); it becomes the position vector $(1.0x - 0.9y, 0.3x + 1.5y)$.

So, thanks to the multiplication rule for matrices, we can say that the effect of the particular linear transformation of the plane that was shown in Figure 31 is to transform any point represented by a 2×1 column vector $\mathbf{x}$ into the point represented by the 2×1 matrix product $\mathbf{Ax}$, where $\mathbf{A} = \begin{bmatrix} 1.0 & -0.9 \\ 0.3 & 1.5 \end{bmatrix}$.

Though this is only one example, you should not be surprised by the following general rule.

Any **linear transformation** of the plane can be represented by a 2×2 matrix $\mathbf{A}$. The effect of the linear transformation on a position vector represented by the 2×1 column vector $\mathbf{x}$ is to transform it into the position vector represented by the 2×1 matrix product $\mathbf{Ax}$.

Of course, all this is deliberately ponderous for the sake of clarity. Those familiar with such transformations usually just say '$\mathbf{A}$ transforms $\mathbf{x}$ to $\mathbf{Ax}$'. Indeed, mathematicians often prefer to describe the effect of the linear transformation in terms of the 'mapping' of vectors rather than their transformation, and will generally say '$\mathbf{A}$ maps $\mathbf{x}$ to $\mathbf{Ax}$'. The column vector $\mathbf{Ax}$ is then described as the *image* of x under the *mapping*

represented by $\mathbf{A}$. The implication is the same, whichever form of language is used: 2×2 matrices can be used to represent linear transformations of the plane, and to work out their effects on position vectors.

Example 11

What is the effect of the linear transformation represented by $\mathbf{A} = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$ on a point with coordinates $(2, 1)$?

Solution

First, we note that a point with coordinates $(2, 1)$ can be represented by the position column vector $\mathbf{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

The effect of the transformation on $\mathbf{x}$ is then given by

$$\mathbf{Ax} = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \times 2 + 2 \times 1 \\ 1 \times 2 + 4 \times 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \end{bmatrix}.$$

So the point $(2, 1)$ is transformed to the point $(8, 6)$.

Exercise 22

Consider the linear transformation represented by $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

Use the matrix $\mathbf{A}$ to find the effect of the transformation on each of the following position column vectors.

(a) $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ (b) $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ (c) $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

We have now achieved our aim of showing how linear transformations of the plane can be represented using matrices. In the next subsection we consider matrix representations of some specific linear transformations with easily visualised actions.

3.2 Some linear transformations of the plane

In this section we examine some specific transformations of the plane. Identifying the 2×2 matrix

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

associated with a particular transformation of the plane is most easily accomplished by examining the effect of the matrix on the unit column vectors. To see the reason for this, consider the matrix products

$$\mathbf{Ai} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix}, \quad \mathbf{Aj} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix}.$$

These show that the transformation represented by the matrix $\mathbf{A}$ is the one that maps $\mathbf{i} = (1, 0)$ to (a, c), and also maps $\mathbf{j} = (0, 1)$ to (b, d). This leads to the following general principle.

> The square matrix $\mathbf{A}$ describing a given linear transformation has columns that are identical to the column vectors produced by the action of the transformation on the unit column vectors.

So if you know what the transformation does to the unit vectors, then you can write down its matrix. The use of this principle is shown in the following example.

Example 12

The matrix $\mathbf{D} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ is said to represent a **rescaling** (or **dilation**) of the plane by a factor of 2 in the positive y-direction.

(a) Work out the effect of this transformation on the point with position vector $(1, 2)$, and the point with position vector $(2, 1)$.

(b) Write down the effect of the transformation on the unit vectors $(1, 0)$ and $(0, 1)$, and justify your answer.

(c) Sketch a diagram to show the general effect of the transformation on the plane and the unit square in Figure 30. Comment on the appropriateness of the description of its action.

Solution

(a) The matrix product

$$\mathbf{D}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

shows that the position vector $(1, 2)$ is mapped to $(1, 4)$.

The matrix product

$$\mathbf{D}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

shows that the position vector $(2, 1)$ is mapped to $(2, 2)$.

(b) $(1, 0)$ is mapped to $(1, 0)$, and $(0, 1)$ is mapped to $(0, 2)$. This is in agreement with the comments made above. The unit vectors $(1, 0)$ and $(0, 1)$ are represented in matrix notation by the unit column vectors. The result of multiplying these unit column vectors by $\mathbf{D}$ is to produce the columns of $\mathbf{D}$ itself.

(c) Figure 33 shows the effect of the transformation. It stretches the plane by a factor of 2 in the positive y-direction, making the description of its action appropriate.

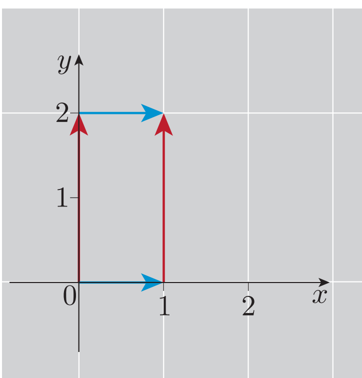


Figure 33 The rescaling of the plane by a factor of 2 in the positive y-direction

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The general rescaling of the plane allows the x-coordinate of every point to be multiplied by a real number κ, while the y-coordinate is multiplied by λ. Such a transformation is shown in Figure 34 and is represented by the following matrix.

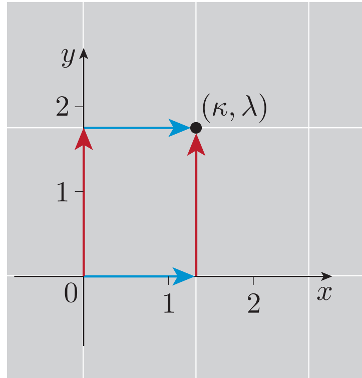

Figure 34 The general rescaling of the plane by κ in the x-direction and λ in the y-direction

The two-dimensional dilation matrix

$$\mathbf{D}(\kappa, \lambda) = \begin{bmatrix} \kappa & 0 \\ 0 & \lambda \end{bmatrix}. \tag{40}$$

Exercise 23

The matrix $\mathbf{D}(3, 2) = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$ represents a rescaling of the plane.

(a) Write down the effect of the transformation on the unit vectors $(1, 0)$ and $(0, 1)$.

(b) Work out the effect of this transformation on the point with coordinates $x = 3$ and $y = -2$.

(c) What is the effect of the transformation on the area of a unit square (i.e. a square that may be edged by unit vectors)?

Rotations about the origin constitute an important class of linear transformations, often encountered in science and engineering. The effect of a rotation about the origin by an angle α in the positive (i.e. anticlockwise) sense is shown in Figure 35. Such a transformation of the plane can be represented by the following matrix.

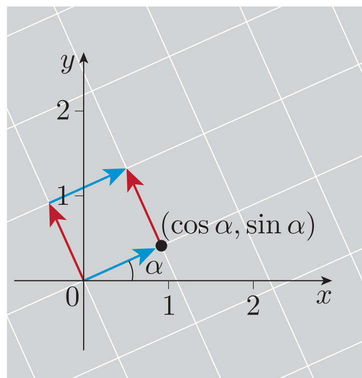

Figure 35 The rotation of the plane represented by $\mathbf{R}(\alpha)$

The two-dimensional rotation matrix

$$\mathbf{R}(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}. \tag{41}$$

Exercise 24

(a) Recalling that $\sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$, write down the matrix that represents a rotation about the origin by $\frac{\pi}{4}$ radians in the anticlockwise sense. (Be explicit, i.e. replace trigonometric functions by arithmetical quantities.)

(b) Write down the effect of this transformation on the unit vectors $(1, 0)$ and $(0, 1)$.

(c) Work out the effect of this transformation on the point with coordinates $x = -1$ and $y = -1$.

(d) What is the effect of the transformation on the area of a unit square?

We end with one of the simplest transformations of the plane: the **identity** transformation. This is the transformation that leaves everything where it was. It is represented by the following matrix.

The two-dimensional identity matrix

$$\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{42}$$

Exercise 25

The identity matrix $\mathbf{I}$ can arise as a special case of the dilation matrix $\mathbf{D}(\kappa, \lambda)$ or the rotation matrix $\mathbf{R}(\alpha)$. For what values of κ, λ and α will this happen?

3.3 Basic matrix algebra and successive transformations

Now that several different matrices have been studied, we can start to develop some of the ideas of basic matrix algebra. For now we restrict the detailed discussion to two dimensions, so that the concrete setting of transformations of the plane can continue to be used. A more general treatment will be provided in Section 4.

Note that throughout this subsection we will be making extensive use of subscripts to distinguish matrix elements; in particular, we will write

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}.$$

Scaling and adding matrices

We start by defining the scaling and adding of matrices, both of which have very natural definitions.

Given a matrix $\mathbf{A}$ of any order, the operation of **scaling** the matrix by a scalar λ produces a matrix $\lambda\mathbf{A}$, of the same order, in which each element is multiplied by λ. Thus, in the case of a 2×2 matrix,

$$\lambda\mathbf{A} = \lambda \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \lambda a_{11} & \lambda a_{12} \\ \lambda a_{21} & \lambda a_{22} \end{bmatrix}. \tag{43}$$

Given two matrices $\mathbf{A}$ and $\mathbf{B}$ of the same order, the operation of **adding** the two matrices produces a matrix $\mathbf{A} + \mathbf{B}$, of the same order, in which each element is the sum of the corresponding elements of $\mathbf{A}$ and $\mathbf{B}$.

Thus, in the case of 2×2 matrices,

$$\mathbf{A} + \mathbf{B} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{bmatrix}. \qquad (44)$$

As you saw in Section 1, a position vector $\mathbf{r}$ may be written as a linear combination of unit vectors. We can now use the scaling and adding of matrices, together with the unit column vectors $\mathbf{i}$ and $\mathbf{j}$, to provide a similar way of writing a position column vector $\mathbf{x}$.

Example 13

Write the position column vector $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ as a linear combination of the unit column vectors $\mathbf{i}$ and $\mathbf{j}$.

Solution

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \end{bmatrix} = x\mathbf{i} + y\mathbf{j}.$$

Exercise 26

Evaluate the result of the following scalings and additions.

(a) $3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} - 2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ (b) $4 \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} - 3 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right)$ (c) $2 \begin{bmatrix} 1 & 0 \\ -2 & 3 \end{bmatrix}$

(d) $2 \left(\begin{bmatrix} 1 & 0 \\ -2 & 3 \end{bmatrix} + 2 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right)$ (e) $2 \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} + 2 \begin{bmatrix} a & 2b \\ -2c & d \end{bmatrix} \right)$

Multiplying matrices

You have already had plenty of practice at multiplying 2×2 square matrices and 2×1 column matrices. Now we extend matrix multiplication to the case of multiplying one 2×2 matrix by another 2×2 matrix. The result is always a 2×2 matrix, and the four elements in the product are worked out using a straightforward extension to the method used earlier. Formally, we can define the product of two 2×2 matrices as follows.

The product of a 2×2 matrix $\mathbf{A}$ and a 2×2 matrix $\mathbf{B}$ is another 2×2 matrix given by

$$\begin{aligned} \mathbf{AB} &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \\ &= \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}. \qquad (45) \end{aligned}$$

It is not recommended that you try to remember this equation, but you should try to remember the process by which the four elements in the result are determined from the elements of $\mathbf{A}$ and $\mathbf{B}$. This may be described by saying that the element in the ith row and the jth column of the product is the sum of the element-by-element products of the ith row of the first matrix and the jth column of the second matrix. In Figure 36 this is illustrated by the hand symbols for the case of the element $a_{11}b_{12} + a_{12}b_{22}$ in the first row and the second column of the product. All the other elements may be worked out in a similar way.

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$$

Figure 36 Obtaining the element in the first row and the second column of the product of two matrices

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Exercise 27

Work out the following matrix products using the method of Figure 36.

(a) $\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 3 \\ 1 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & -3 \end{bmatrix}$

Successive transformations

Now consider what happens when we carry out one transformation of the plane and then perform another transformation on the result – e.g. a rescaling followed by a rotation. Let us represent the first transformation by the dilation matrix $\mathbf{D}(\kappa, \lambda)$ given in equation (40), and the second by the rotation matrix $\mathbf{R}(\alpha)$ given in equation (41). For the sake of definiteness, consider their action on a specific point initially located at the position represented by the column vector $\mathbf{x}_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$. This is indicated schematically in Figure 37.

$$\mathbf{x}_0 \xrightarrow{\mathbf{D}(\kappa, \lambda)} \mathbf{x}_1 \xrightarrow{\mathbf{R}(\alpha)} \mathbf{x}_2$$

Figure 37 The effect of successive transformations on $\mathbf{x}_0$

Suppose that the first transformation, the rescaling, transforms $\mathbf{x}_0$ into the position column vector $\mathbf{x}_1 = \mathbf{D}(\kappa, \lambda)\,\mathbf{x}_0$, so that

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} \kappa & 0 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} \kappa x_0 \\ \lambda y_0 \end{bmatrix}. \tag{46}$$

Also suppose that the second transformation, the rotation, then transforms $\mathbf{x}_1$ into the position column vector $\mathbf{x}_2 = \mathbf{R}(\alpha)\,\mathbf{x}_1$, so that

Note that when multiplying factors such as $\cos\alpha$ and x_1, we prefer to write the result as $x_1 \cos\alpha$ since the alternative $\cos\alpha x_1$ could be misinterpreted as $\cos(\alpha x_1)$.

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} x_1 \cos\alpha - y_1 \sin\alpha \\ x_1 \sin\alpha + y_1 \cos\alpha \end{bmatrix}.$$

But from equation (46) we have $x_1 = \kappa x_0$ and $y_1 = \lambda y_0$, so we can rewrite this result as

$$\mathbf{x}_2 = \mathbf{R}(\alpha)\,\mathbf{D}(\kappa, \lambda)\,\mathbf{x}_0 = \begin{bmatrix} \kappa x_0 \cos\alpha - \lambda y_0 \sin\alpha \\ \kappa x_0 \sin\alpha + \lambda y_0 \cos\alpha \end{bmatrix}.$$

Now, this is significant because it has the same general form as the result of applying a single 2×2 matrix, $\mathbf{C}$ say, to $\mathbf{x}_0$. The effect of $\mathbf{C}$ is indicated schematically in Figure 38. If our definition of matrix multiplication makes sense, we should expect the matrix $\mathbf{C}$ to be the product $\mathbf{R}(\alpha)\,\mathbf{D}(\kappa,\lambda)$. Let us check to see if this is correct. First, let

$$\mathbf{C} = \mathbf{R}(\alpha)\,\mathbf{D}(\kappa,\lambda) = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \kappa & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} \kappa\cos\alpha & -\lambda\sin\alpha \\ \kappa\sin\alpha & \lambda\cos\alpha \end{bmatrix}.$$

Then note that

$$\mathbf{C}\mathbf{x}_0 = \begin{bmatrix} \kappa\cos\alpha & -\lambda\sin\alpha \\ \kappa\sin\alpha & \lambda\cos\alpha \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} \kappa x_0\cos\alpha - \lambda y_0\sin\alpha \\ \kappa x_0\sin\alpha + \lambda y_0\cos\alpha \end{bmatrix} = \mathbf{x}_2.$$

This confirms our expectations. The conclusion is clear and can be stated in general terms as follows.

$$\begin{tikzcd} \mathbf{x}_0 \arrow[r, "\mathbf{D}(\kappa,\lambda)"] \arrow[rr, bend right, "\mathbf{C}"'] & \mathbf{x}_1 \arrow[r, "\mathbf{R}(\alpha)"] & \mathbf{x}_2 \end{tikzcd}$$

Figure 38 The effect of the combined transformation $\mathbf{C}$ on $\mathbf{x}_0$

> The effect on a column vector $\mathbf{x}$ of a first transformation, represented by $\mathbf{A}$, followed by a second transformation, represented by $\mathbf{B}$, is equivalent to the effect of a single transformation, represented by the matrix product $\mathbf{BA}$.

Pay close attention to the order of the matrices in the above result. The first matrix to act, $\mathbf{A}$, is the one that appears on the *right* in the product $\mathbf{BA}$. This may look a little odd if you are not familiar with matrices, but it has to be so, since we are constructing a product that will act on any column vector that appears even further to the right in the combination $\mathbf{BAx}$.

Exercise 28

Find a matrix $\mathbf{C}$ that can act on a column vector $\mathbf{x}$ to reproduce the effect of a rescaling represented by $\mathbf{D}(2,1)$ followed by a rotation $\mathbf{R}\left(\frac{\pi}{4}\right)$.

In the example above, and in all the work that led up to it, we have been careful to preserve the order of matrices in a matrix product. This is important because matrix multiplication is not generally commutative: $\mathbf{AB}$ may be very different from the product $\mathbf{BA}$.

Exercise 29

Find the matrix $\mathbf{F}$ that represents the product of the transformations in Exercise 28 but in the reverse order, i.e. $\mathbf{F} = \mathbf{D}(2,1)\,\mathbf{R}\left(\frac{\pi}{4}\right)$. Show that, in general, $\mathbf{F}$ is not the same matrix as $\mathbf{C}$, i.e. $\mathbf{D}(2,1)\,\mathbf{R}\left(\frac{\pi}{4}\right) \neq \mathbf{R}\left(\frac{\pi}{4}\right)\mathbf{D}(2,1)$.

It is not really surprising that the result of a rotation $\mathbf{R}\left(\frac{\pi}{4}\right)$ followed by a rescaling $\mathbf{D}(2,1)$ differs from the result of performing those operations in the reverse order. It is symptomatic of the following general rule.

Matrix multiplication is non-commutative

Though there are cases where matrices *do* commute, generally

$$\mathbf{AB} \neq \mathbf{BA}.$$

Despite being non-commutative, matrix multiplication is *associative*, so

$$(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC}). \tag{47}$$

It is also the case that matrix multiplication is *distributive over addition*, so

$$\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}, \tag{48}$$

and matrix multiplication is *linear* with respect to scalar multiplication, so

$$\lambda(\mathbf{AB}) = \mathbf{A}(\lambda\mathbf{B}), \quad \text{where } \lambda \text{ is a scalar.} \tag{49}$$

As usual, we can use these properties to make sense of matrix expressions that involve brackets.

Matrices are ideally suited to describing situations where the order of operations really matters, as in much of science and also in everyday life. Do not think of a matrix inequality such as $\mathbf{AB} \neq \mathbf{BA}$ as something frightful. It might just signify that the operations of putting sugar into tea cups and drinking tea from those cups do not commute: the order in which you do things clearly matters in this situation, and many others.

Despite the generally non-commutative nature of matrix multiplication, there are cases where the matrices *do* commute, so their order may be reversed without changing the result. One such case is when one of the transformations is the identity transformation represented by the identity matrix $\mathbf{I}$ (first introduced in equation (42)). Commutation in this case makes good sense, since the identity transformation doesn't change anything, so you would expect it to have the same lack of effect whether it was done first or last. As you can easily confirm for any 2×2 matrix $\mathbf{A}$,

$$\mathbf{IA} = \mathbf{AI} = \mathbf{A}. \tag{50}$$

Exercise 30

Show that products of rotations in two dimensions do commute by establishing that

$$\mathbf{R}(\alpha)\,\mathbf{R}(\beta) = \mathbf{R}(\beta)\,\mathbf{R}(\alpha) = \mathbf{R}(\alpha + \beta).$$

3.4 Undoing transformations and matrix inversion

Often, after performing an operation such as a linear transformation of the plane, we need to reverse the changes that have been made and undo the transformation. In matrix algebra this is achieved by following the action of a matrix, $\mathbf{A}$ say, by the action of the *inverse matrix*, which is

denoted $\mathbf{A}^{-1}$. The combination of the two, represented by their matrix product, results in no change, so it must be equal to the identity matrix $\mathbf{I}$. The same must be true if we perform the inverse transformation first, and then follow that with the original transformation. Interestingly, inverse transformations do not always exist. However, when they do exist, they are *unique*, so what we can say in general is the following.

Given a square matrix $\mathbf{A}$, its **inverse matrix**, if it exists, is denoted $\mathbf{A}^{-1}$, has the same order as $\mathbf{A}$, and satisfies the condition

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}. \tag{51}$$

Also, if $\mathbf{A}$ and $\mathbf{B}$ are square matrices of the same order, and $\mathbf{A}\mathbf{B} = \mathbf{B}\mathbf{A} = \mathbf{I}$, then $\mathbf{B} = \mathbf{A}^{-1}$.

The process of finding the inverse of a matrix is called **matrix inversion**. In some cases it is easy, even obvious. In other cases it can be difficult or simply impossible. A matrix that can be inverted is said to be **invertible**. In this subsection we examine some simple cases, describe a general procedure for finding the inverse of a 2×2 matrix, and determine the criterion for deciding whether a given 2×2 matrix is invertible. The inversion of larger matrices will be discussed in Section 4.

Simple matrix inversions

The inverse of the identity matrix is the identity matrix itself, i.e. $\mathbf{I}^{-1} = \mathbf{I}$. This is clear, since $\mathbf{I}\,\mathbf{I} = \mathbf{I}$.

The inverse of the dilation matrix $\mathbf{D}(\kappa, \lambda)$ is $\mathbf{D}^{-1}(\kappa, \lambda) = \mathbf{D}(1/\kappa, 1/\lambda)$.

The inverse of the rotation matrix $\mathbf{R}(\alpha)$ is $\mathbf{R}^{-1}(\alpha) = \mathbf{R}(-\alpha)$.

All of the above inverses are guaranteed to exist, apart from the inverse dilation matrix in the special case that $\kappa = 0$ or $\lambda = 0$.

Example 14

Confirm by matrix multiplication that the matrix $\begin{bmatrix} 1/\kappa & 0 \\ 0 & 1/\lambda \end{bmatrix}$ is the inverse of the dilation matrix $\begin{bmatrix} \kappa & 0 \\ 0 & \lambda \end{bmatrix}$, for non-zero κ and λ.

Solution

It is sufficient to note that

$$\begin{bmatrix} 1/\kappa & 0 \\ 0 & 1/\lambda \end{bmatrix} \begin{bmatrix} \kappa & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} \kappa & 0 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} 1/\kappa & 0 \\ 0 & 1/\lambda \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Exercise 31

Confirm by explicit matrix multiplication that the inverse of the rotation matrix $\mathbf{R}(\alpha)$ is $\mathbf{R}(-\alpha)$.

Inversion of a general 2×2 matrix

Consider the matrix product

$$\mathbf{AB} = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}.$$

It is obvious that $\mathbf{A}$ is not the inverse of $\mathbf{B}$, but it is also clear that $\frac{1}{5}\mathbf{A}$ will be the inverse of $\mathbf{B}$. This technique, of first identifying a matrix with the right structure and than scaling it by an appropriate factor, is the basis of the following general rule for finding the inverse of a 2×2 matrix.

Inverse of a 2×2 matrix

Given the 2×2 matrix $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, its inverse matrix, if it exists, is given by

$$\mathbf{A}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}. \tag{52}$$

Example 15

Using the general expressions given above for a 2×2 matrix $\mathbf{A}$ and its inverse $\mathbf{A}^{-1}$, verify by matrix multiplication that $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$.

Solution

$$\frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix} da - bc & db - bd \\ -ca + ac & -cb + ad \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Exercise 32

Using equation (52) again, verify that $\mathbf{AA}^{-1} = \mathbf{I}$.

Exercise 33

Let $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$.

(a) Find the inverses of $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D} = \mathbf{ABC}$.

(b) Verify that $\mathbf{D}^{-1} = \mathbf{C}^{-1}\mathbf{B}^{-1}\mathbf{A}^{-1}$. (In other words, verify that $(\mathbf{ABC})^{-1} = \mathbf{C}^{-1}\mathbf{B}^{-1}\mathbf{A}^{-1}$.)

Note the interesting pattern revealed by this exercise:

$$(\mathbf{ABC})^{-1} = \mathbf{C}^{-1}\mathbf{B}^{-1}\mathbf{A}^{-1}.$$

As you will see in Section 4, this is actually a general result: the inverse of a product of matrices is equal to the product of the inverses *in reverse order*.

Criterion for the existence of an inverse

In equation (52), the general formula for inverting a 2×2 matrix $\mathbf{A}$, it is necessary to divide every term in a matrix by the quantity $ad - bc$. This is mathematically meaningful only if $ad - bc$ is not equal to zero. That is why it is impossible to invert some 2×2 matrices. As you will see later, the quantity $ad - bc$ for the matrix $\mathbf{A}$ is a particularly simple example of a mathematical entity called a *determinant*. Every square matrix $\mathbf{A}$ has a determinant, usually denoted $\det \mathbf{A}$, but only in the case of 2×2 matrices can it be generally expressed as simply as $ad - bc$. Using the determinant we can say the following.

Criterion for the existence of $\mathbf{A}^{-1}$

Given the 2×2 matrix $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, its inverse matrix $\mathbf{A}^{-1}$ exists if and only if

$$\det \mathbf{A} = ad - bc \neq 0. \qquad (53)$$

From the matrix inversion formula equation (52), it is clear why $ad - bc$ is important, but in the case of 2×2 matrices it is possible to get a deeper and mathematically more interesting insight into the origin of the existence criterion. Viewed as a transformation of the plane, the effect of the matrix $\mathbf{A}$ is to map the unit vectors $(1, 0)$ and $(0, 1)$ into the vectors (a, c) and (b, d), as indicated in Figure 39.

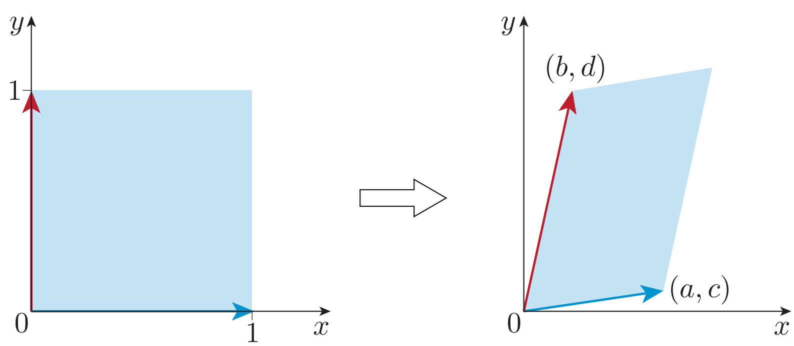


Figure 39 The geometric effect of a general linear transformation of the plane

As a result, a unit square of area 1 is mapped into a parallelogram of area $ad - bc$. (You established that the area of a parallelogram is $|ad - bc|$ in Exercise 19.)

Consequently, if the matrix $\mathbf{A}$ that describes the mapping has $\det \mathbf{A} = 0$, so $ad - bc = 0$, then the area of that parallelogram must also be zero. This means that the corners of the parallelogram must be either on the same line (i.e. collinear), or all at the same point. This is indicated in Figure 40. In this extreme case, the action of $\mathbf{A}$ has been to collapse the unit square to such an extent that there is insufficient information in the resulting 'parallelogram' (actually either a line or a point) to allow the original unit square to be reconstructed by the inverse transformation $\mathbf{A}^{-1}$. That is why the inverse transformation cannot exist.

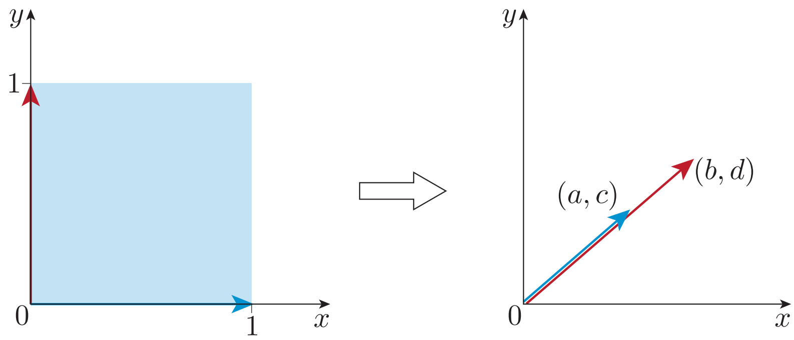


Figure 40 The geometric effect of a linear transformation with zero determinant; the parallelogram is reduced to a line or a point

Exercise 34

Which of the following matrices has no inverse?

$$\begin{bmatrix} 1 & -2 \\ 2 & 4 \end{bmatrix}, \quad \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix}, \quad \begin{bmatrix} 4 & 1 \\ 2 & -2 \end{bmatrix}, \quad \begin{bmatrix} -2 & 1 \\ 4 & 2 \end{bmatrix}.$$

Exercise 35

Referring to the three matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ introduced in Exercise 33, calculate the determinants of $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{ABC}$. Verify that $\det(\mathbf{ABC}) = \det \mathbf{A} \det \mathbf{B} \det \mathbf{C}$.

Exercise 36

The determinant of

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$$

vanishes. What is the effect of this transformation on the Cartesian unit vectors? Does $\mathbf{A}$ indeed transform the unit square to a geometric object with no area?

4 Matrix algebra

Section 3 was concerned with examples of matrix action that were easy to visualise, so the discussion was mostly confined to 2×2 matrices that could be interpreted geometrically. Such transformations are important, but they are only one indication of the great wealth of applications of matrix algebra in general. There are many applications of matrices that do not involve geometry, and many that involve matrices larger than 2×2.

An example from biology

As a brief example, consider a biologist studying the effects of nutrition on a group of animals. Suppose that two nutrients n_1 and n_2 are being fed to the animals in two different foodstuffs f_1 and f_2. Also suppose that the mass of nutrient n_i per kilogram of foodstuff f_j is a_{ij}. The situation is depicted schematically in Figure 41.

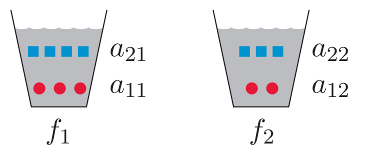


Figure 41 Foodstuff f_1 contains a quantity a_{11} of nutrient n_1 (red circle), and a_{21} of nutrient n_2 (blue square); similarly for foodstuff f_2

A feeding scheme that supplies each animal with a mass M_j of foodstuff f_j will automatically supply each animal with a mass m_i of nutrient n_i given by $m_i = a_{i1}M_1 + a_{i2}M_2$. You should recognise this as the ith element in the matrix

$$\begin{bmatrix} m_1 \\ m_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} M_1 \\ M_2 \end{bmatrix}. \tag{54}$$

It would take quite a lot of effort to use matrix mathematics for such a simple problem. Suppose, however, that the biologist is actually interested in 20 nutrients being delivered in different proportions by 20 foodstuffs. The delivery of nutrients by a given combination of foodstuffs would then be described by the product of a 20×20 matrix $\mathbf{A}$ and a 20×1 feeding matrix $\mathbf{M}$. Matrix multiplication would provide a useful tool for keeping track of such a complicated arrangement, which would be described by a system of 20 equations. In addition, the problem of finding the combination of foodstuffs that ensures the delivery of the required amounts of each nutrient might be tackled by determining the inverse of the 20×20 matrix.

Similar (less contrived) problems arise every day in a range of activities. As a result, the study of the **matrix algebra** of general $m \times n$ matrices is an important part of almost every course in higher mathematics, pure or applied. In addition, the algebraic principles that it teaches extend beyond the domain of matrices, rich as that is. In particular, matrix algebra is now recognised as the principal example of a broader subject known as *linear algebra*, which will be introduced in Unit 5.

Many aspects of 2×2 matrix algebra were covered in Section 3. In this section we generalise to the case of $m \times n$ matrices.

4.1 Notation and fundamentals of matrix algebra

As usual, we represent a matrix of order $m \times n$ (i.e. m rows and n columns) by a bold symbol, such as $\mathbf{A}$, with a_{ij} representing the element in its ith row and jth column. In an extension of our earlier notation we henceforth use the notation $[a_{ij}]$ to indicate the whole matrix of elements a_{ij}. So writing $\mathbf{A} = [a_{ij}]$ means exactly the same as equation (38). Similarly, we write $\mathbf{B} = [b_{ij}]$ for a matrix $\mathbf{B}$ of elements b_{ij}, etc.

Concepts such as equating matrices, adding matrices (of the same order) and scaling matrices are all easily generalised to the case of $m \times n$ matrices. We will state the rules describing them later, but there will be no surprises. Slightly more challenging is the generalisation of matrix multiplication, so we deal with that first.

The first thing to remember about the general case of matrix multiplication is that it is possible only when the matrices involved are of the appropriate orders. As indicated by the hand symbols in Figures 32 and 36, matrix multiplication involves summing the products of corresponding elements from a row of the first matrix and a column of the second matrix. This is possible only if the number of columns in the first matrix is equal to the number of rows in the second. This is embodied in the following rule concerning the order of matrix products.

Rule for the existence and order of a matrix product

An $m \times q$ matrix $\mathbf{A}$ may be multiplied by an $r \times n$ matrix $\mathbf{B}$ to form the matrix product $\mathbf{AB}$ if and only if $q = r$, in which case the product $\mathbf{AB}$ will be of order $m \times n$.

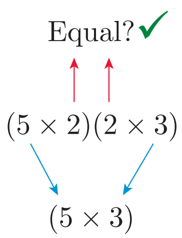


Figure 42 A visual reminder of the rule for the existence and order of a matrix product. Here $\mathbf{A}$ has order 5×2, $\mathbf{B}$ has order 2×3, and $\mathbf{AB}$ has order 5×3.

Because of this rule, when considering multiplying $\mathbf{A}$ by $\mathbf{B}$, you may find it helpful to picture something like Figure 42, with the order of $\mathbf{A}$ written to the left of the order of $\mathbf{B}$. This will produce a row of four integers. If the two inner integers are equal, the matrices can be multiplied together. In such cases, the order of the matrix product $\mathbf{AB}$ is given by the two outer integers.

Supposing that the matrices $\mathbf{A}$ and $\mathbf{B}$ are suitable for multiplication, how should the result be determined? The answer is based on a straightforward generalisation of the method used for multiplying 2×2 matrices in Section 3. It is embodied in the following rule.

Rule for the evaluation of a matrix product

The element in the ith row and jth column of the matrix product $\mathbf{AB}$ is the sum of the element-by-element products of the ith row of $\mathbf{A}$ with the jth column of $\mathbf{B}$.

This way of combining rows and columns should already be familiar from earlier examples, but the whole idea is illustrated again, schematically, in Figure 43.

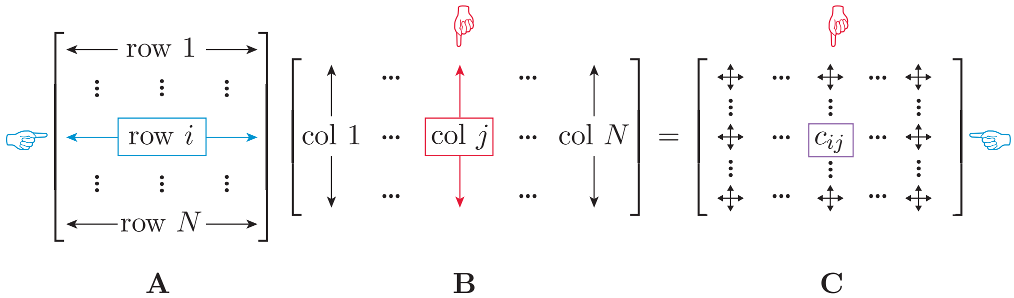


Figure 43 A visual prompt to help you to remember the rule for the evaluation of a matrix product

The consequence of the rule can be represented formally (but less helpfully) by saying that when an $m \times r$ matrix $\mathbf{A}$ multiplies an $r \times n$ matrix $\mathbf{B}$ to form the $m \times n$ matrix product $\mathbf{AB} = \mathbf{C}$, the element c_{ij} in the ith row and jth column of $\mathbf{C}$ is given by

Don't try to remember this formula. Remember instead the method shown in Figure 43 that underlies it.

$$c_{ij} = \sum_{k=1}^{r} a_{ik}b_{kj} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ir}b_{rj}. \tag{55}$$

Example 16

Consider the matrix product

$$\begin{bmatrix} 2 & -1 & 3 \\ -2 & 1 & 2 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ 2 & 2 \\ -2 & -2 \end{bmatrix}.$$

Write down the order of the resulting matrix, and evaluate the product.

Solution

The product of a 2×3 matrix and a 3×2 matrix is a 2×2 matrix:

$$\begin{aligned}
&\begin{bmatrix} 2 & -1 & 3 \\ -2 & 1 & 2 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ 2 & 2 \\ -2 & -2 \end{bmatrix} \\
&= \begin{bmatrix} 2(3) - 1(2) + 3(-2) & 2(-1) - 1(2) + 3(-2) \\ -2(3) + 1(2) + 2(-2) & -2(-1) + 1(2) + 2(-2) \end{bmatrix} \\
&= \begin{bmatrix} 6 - 2 - 6 & -2 - 2 - 6 \\ -6 + 2 - 4 & 2 + 2 - 4 \end{bmatrix} \\
&= \begin{bmatrix} -2 & -10 \\ -8 & 0 \end{bmatrix}.
\end{aligned}$$

Exercise 37

Evaluate the following matrix products, where they exist.

(a) $\begin{bmatrix} 1 & 2 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 6 \\ 0 & 2 \end{bmatrix}$ (c) $\begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 3 & 0 & -4 \end{bmatrix}$

(d) $\begin{bmatrix} 3 & 1 & 2 \\ 0 & 5 & 1 \end{bmatrix} \begin{bmatrix} -2 & 0 & 1 \\ 1 & 3 & 0 \\ 4 & 1 & -1 \end{bmatrix}$ (e) $\begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 4 & -1 \end{bmatrix}$

As you saw in Section 3, matrix multiplication is *associative* (so $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$) and *distributive over matrix addition* (so $\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{AB} + \mathbf{AC}$). Matrix multiplication is also *linear* with regard to scalar multiplication (so $\mathbf{A}\lambda\mathbf{B} = \lambda\mathbf{AB}$). However, matrix multiplication is *not* generally commutative. This last point is so important that it deserves its own box, even though you have seen it before.

In general,

$$\mathbf{AB} \neq \mathbf{BA},$$

although there are cases where matrices *do* commute

Exercise 38

Contrast this with rotations in two dimensions, which do commute, as we showed in Exercise 30.

In three-dimensional space, rotations do not, in general, commute. For example, consider the matrices

$$\mathbf{A} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$

The first matrix transforms space by rotation of $\frac{\pi}{2}$ about the z-axis; the second matrix transforms space by rotation of $\frac{\pi}{2}$ about the x-axis.

Let $\mathbf{C}$ be the transformation of performing $\mathbf{A}$ and then $\mathbf{B}$; let $\mathbf{D}$ be the transformation of performing $\mathbf{B}$ and then $\mathbf{A}$. Calculate the matrices $\mathbf{C}$ and $\mathbf{D}$, and hence decide if $\mathbf{A}$ and $\mathbf{B}$ commute.

Transposing a matrix

In another extension to our earlier notation, we introduce the operation of taking the *transpose*, which interchanges the rows and columns of a matrix, so that the first row of a matrix becomes the first column of the transposed matrix, and so on. This operation is indicated by a superscript T, so we can write

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 & 7 \\ -6 & 1 \\ 0 & 4 \end{bmatrix}^T = \begin{bmatrix} 2 & -6 & 0 \\ 7 & 1 & 4 \end{bmatrix}.$$

This useful operation has many applications, including allowing us to save space by writing potentially long column matrices as transposed row matrices, as in $\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T$.

Using our compact notation for matrices, with $\mathbf{A} = [a_{ij}]$, we can define the transpose as follows.

Given any matrix $\mathbf{A}$, its **transpose** $\mathbf{A}^T$ is defined by $[a_{ij}]^T = [a_{ji}]$.

$\mathbf{A}^T$ is read as 'A transpose'.

Example 17

Show that in the case that $\mathbf{A}$ is a 2×2 matrix and $\mathbf{x}$ is a 2×1 matrix, it will be always be the case that $(\mathbf{Ax})^T = \mathbf{x}^T\mathbf{A}^T$.

Solution

Let $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$. Then

$$\mathbf{Ax} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix},$$

so

$$(\mathbf{Ax})^T = \begin{bmatrix} ax + by & cx + dy \end{bmatrix}.$$

But

$$\mathbf{x}^T\mathbf{A}^T = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} xa + yb & xc + yd \end{bmatrix}.$$

So it is true that $(\mathbf{Ax})^T = \mathbf{x}^T\mathbf{A}^T$ in this case.

In fact, this example illustrates the following general result concerning the transposition of matrix products.

If the matrix product $\mathbf{AB}$ exists, then its transpose $(\mathbf{AB})^T$ is given by

$$(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T. \tag{56}$$

Notice the reversed sequence of the matrices: the transpose of a product is the product of the transposes *in reverse order*.

General rules of matrix algebra

We are now in a position to summarise the general rules of matrix algebra, including those for scaling and adding that were promised earlier. In summarising the rules, we again use the compact notation in which $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{ij}]$.

Basic rules of matrix algebra

- Two matrices $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{ij}]$ are **equal** if they have the same order $m \times n$, and

  $$a_{ij} = b_{ij} \quad \text{for all } i = 1, 2, \ldots, m \text{ and } j = 1, 2, \ldots, n.$$

- If $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{ij}]$ are any two matrices of the same order, then their **matrix sum** is defined by

  $$\mathbf{A} + \mathbf{B} = [a_{ij} + b_{ij}] \quad \text{(i.e. element-wise addition).}$$

- If $\mathbf{A} = [a_{ij}]$ is any matrix, then there exists a **zero matrix 0**, of the same order, such that

  $$\mathbf{A} + \mathbf{0} = \mathbf{A} \quad \text{(i.e. the elements are unaltered by adding 0).}$$

- If $\mathbf{A} = [a_{ij}]$ is any matrix and k is any scalar, then the **scalar multiplication** of $\mathbf{A}$ by k is defined by

  $$k\mathbf{A} = [ka_{ij}] \quad \text{(i.e. element-wise multiplication by } k\text{).}$$

- If $\mathbf{A} = [a_{ij}]$ is a matrix of order $m \times r$ and $\mathbf{B} = [b_{ij}]$ is a matrix of order $r \times n$, then the **matrix product** $\mathbf{C} = \mathbf{AB}$ exists and is a matrix of order $m \times n$, where the element c_{ij} is defined by

  $$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ir}b_{rj}.$$

- If $\mathbf{A} = [a_{ij}]$ is any matrix of order $m \times n$, then there exists an **identity matrix I** of order $m \times m$ such that

  $$\mathbf{IA} = \mathbf{A},$$

  and there also exists an identity matrix of order $n \times n$, also denoted $\mathbf{I}$, such that

  $$\mathbf{AI} = \mathbf{A}.$$

- If $\mathbf{A} = [a_{ij}]$ is a matrix of order $m \times n$, then its **transpose** $\mathbf{A}^T$ is a matrix of order $n \times m$ defined by

  $$[a_{ij}]^T = [a_{ji}] \quad \text{(i.e. the interchange of rows and columns).}$$

- If the matrix product $\mathbf{AB}$ exists, then the **transposed product** $(\mathbf{AB})^T$ also exists, and is given by

  $$(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T \quad \text{(note the reversed order).}$$

Additionally, note the following.

- The **subtraction** of a matrix should be interpreted as the addition of a matrix that has been multiplied by the scalar -1.
- The **power** of a matrix should be interpreted as repeated multiplication, as in $\mathbf{A}^2 = \mathbf{AA}$, $\mathbf{A}^3 = \mathbf{AAA}$, etc.
- Expressions involving *brackets* should be interpreted in the usual way, though the ordering of products of matrices should not be changed since matrix multiplication is generally **non-commutative**.

Exercise 39

Let $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 2 & 5 \\ -1 & -4 \\ 3 & 1 \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$.

(a) Write down $\mathbf{A}^T$, $\mathbf{B}^T$ and $\mathbf{C}^T$.

(b) Verify that $(\mathbf{A} + \mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$.

(c) Verify that $(\mathbf{AC})^T = \mathbf{C}^T\mathbf{A}^T$.

4.2 Determinants, inverses and matrix algebra

Determinants

The term *determinant* was introduced into mathematics in 1801 by Carl Friedrich Gauss (1777–1855), though the idea can be traced back to the eighteenth century and beyond. (You will learn more about Gauss in Unit 5, when we discuss a technique known as *Gaussian elimination*.) A 'determinant' was originally associated with a general square array of numbers. However, following the introduction of matrices in 1850, by the British mathematician James Sylvester (1814–1897), and the development of matrix theory by Sylvester's friend and colleague Arthur Cayley (1821–1895), the meaning of 'determinant' became strongly associated with square matrices, and we now generally speak of the 'determinant of a (square) matrix'. This is the spirit in which we introduced determinants in Section 3, where we said that the determinant of the 2×2 square matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ was the quantity $ad - bc$. More generally, according to this modern view, we have the following.

The **determinant** of any square matrix $\mathbf{A}$ is a single value, denoted $\det \mathbf{A}$, that may be calculated from the elements of $\mathbf{A}$ by following a standard prescription.

The 'standard prescription' for evaluating determinants can be expressed in various ways. The form that we use is called *Laplace's rule*, and will be stated later. First, however, let us look at some examples of determinants, to see what Laplace's rule must achieve.

When using the elements of a square matrix to calculate a determinant, it is conventional to write those elements between vertical lines (a practice introduced by Cayley in 1853). Thus if $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we can write

$$\det \mathbf{A} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc. \qquad (57)$$

This convention allows us to speak of the 'rows' and 'columns' of a determinant, and to describe a determinant with n rows and n columns as an $n \times n$ determinant, even though, when finally evaluated, the determinant is only a single value.

Determinants may be associated with square matrices of any order. However, in practice a very important case is that of a 3×3 matrix. As you will see later, Laplace's rule tells us that the determinant of such a matrix may be written as follows.

Determinant of a 3×3 matrix

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}. \qquad (58)$$

At present we ask you to simply accept this formula. Shortly, you will learn a simple rule that enables you to construct the determinant of any $n \times n$ matrix.

Having expressed the 3×3 determinant as a linear combination of 2×2 determinants, we can use equation (57) to work out the 2×2 determinants and thus complete the evaluation. You can see this in the following example.

Example 18

If $\mathbf{A} = \begin{bmatrix} 2 & 2 & -1 \\ 3 & 5 & 1 \\ 1 & 2 & 1 \end{bmatrix}$, evaluate $\det \mathbf{A}$.

Solution

Using equation (58),

$$\begin{aligned} \det \mathbf{A} &= 2 \begin{vmatrix} 5 & 1 \\ 2 & 1 \end{vmatrix} - 2 \begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} - \begin{vmatrix} 3 & 5 \\ 1 & 2 \end{vmatrix} \\ &= 2 \times (5-2) - 2 \times (3-1) - (6-5) \\ &= 6 - 4 - 1 \\ &= 1. \end{aligned}$$

Exercise 40

Evaluate the determinants of the following.

(a) $\mathbf{A} = \begin{bmatrix} 2 & -1 & 2 \\ 5 & 1 & -1 \\ 2 & 1 & -1 \end{bmatrix}$ (b) $\mathbf{B} = \begin{bmatrix} 0 & 2 & -1 \\ 3 & 0 & -1 \\ 2 & -1 & 0 \end{bmatrix}$

Examining these examples of 3×3 determinant expansions will help you to make sense of the expansion of determinants in general. The expression on the right-hand side of equation (58) is the sum of three terms. Each of those terms involves a 2×2 determinant. The origin of those 2×2 determinants is shown in Figure 44.

Element a_{1j}	$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$	$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$	$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$
Minor M_{1j}	$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$	$M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$	$M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$
Cofactor $C_{1j} = (-1)^{1+j} M_{1j}$	$C_{11} = +M_{11}$	$C_{12} = -M_{12}$	$C_{13} = +M_{13}$

Figure 44 Expanding a determinant using the first row of elements and their cofactors

As indicated, the first of the 2×2 determinants is obtained from the original 3×3 determinant by deleting all the elements in the same row and column as the element a_{11} and forming the determinant of what remains. This 2×2 determinant is called the *minor* of the element a_{11}, and is denoted M_{11}. In a similar way, the other two 2×2 determinants in Figure 44 are the minors M_{12} and M_{13} of a_{12} and a_{13}, respectively. We can define the **minor** M_{ij} of any element a_{ij} of a determinant in a similar way, by deleting row i and column j, and forming the determinant of what remains.

To obtain the entire expression on the right of equation (58), each of the relevant minors M_{ij} must first be multiplied by the factor $(-1)^{i+j}$ to produce $C_{ij} = (-1)^{i+j} M_{ij}$, which is known as the **cofactor** of a_{ij}. (If $i + j$ is even, then $C_{ij} = M_{ij}$, but if $i + j$ is odd, then $C_{ij} = -M_{ij}$.) Finally, we must multiply each of the relevant cofactors by the corresponding element a_{ij}, and add the resulting products. In the case of equation (58) this gives us $a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$, which reproduces the right-hand side of the equation, including the alternating signs. The whole process is described as 'the expansion of the 3×3 determinant by the cofactors of its first row of elements'.

Laplace's rule for expanding (and hence evaluating) determinants is a simple generalisation of the procedure that has just been described, which you have already seen in Example 18 and carried out in Exercise 40. The main difference is that the rule tells us how to perform the expansion based on *any* row or column, and applies to a determinant of *any* $n \times n$ matrix. The rule may be stated as follows.

Although we are free to use any row or column when applying Laplace's rule, it is usual to use the first row.

Laplace's rule for expanding determinants

Given an $n \times n$ matrix $\mathbf{A} = [a_{ij}]$, its determinant $\det \mathbf{A}$ may be expanded in terms of the elements in row i and their cofactors C_{ij} as

$$\det \mathbf{A} = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}, \tag{59}$$

or equivalently, in terms of the elements in column j and their cofactors C_{ij}, as

$$\det \mathbf{A} = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}, \tag{60}$$

where $C_{ij} = (-1)^{i+j}M_{ij}$, and M_{ij} is the minor obtained by deleting row i and column j of the original determinant and forming the determinant of what remains.

For obvious reasons, Laplace's rule is sometimes called the **cofactor expansion rule**.

Example 19

Evaluate $\det \mathbf{A} = \begin{vmatrix} 1 & 0 & 0 & 3 \\ 2 & 2 & -1 & 2 \\ 3 & 5 & 1 & -1 \\ 1 & 2 & 1 & -1 \end{vmatrix}$.

Solution

To simplify the evaluation, we should expand using the first row, since that contains two zero elements, so it will minimise the work required. Thus

$$\begin{aligned} \det \mathbf{A} &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} + a_{14}C_{14} \\ &= C_{11} + 3C_{14}. \end{aligned}$$

Now $C_{11} = (-1)^{1+1}M_{11} = M_{11}$, where M_{11} is the determinant of the matrix obtained by crossing out the first row and first column of $\mathbf{A}$, i.e.

$$C_{11} = M_{11} = \begin{vmatrix} 2 & -1 & 2 \\ 5 & 1 & -1 \\ 2 & 1 & -1 \end{vmatrix}.$$

This determinant was evaluated in Exercise 40, where we obtained the result

$$C_{11} = 3.$$

Further, $C_{14} = (-1)^{1+4}M_{11} = -M_{11}$, where M_{11} is the determinant of the matrix obtained by crossing out the first row and fourth column of $\mathbf{A}$, i.e.

$$C_{14} = -M_{14} = -\begin{vmatrix} 2 & 2 & -1 \\ 3 & 5 & 1 \\ 1 & 2 & 1 \end{vmatrix} = -1,$$

where we have used the result obtained in Example 18. Hence

$$\det \mathbf{A} = 3 - 3 = 0.$$

The evaluation of determinants can be very time consuming, so here are some rules that exploit the intrinsic symmetry of determinants to speed up the process.

Rules for the determinant of an $n \times n$ matrix A

- Interchanging any two rows or any two columns of $\mathbf{A}$ changes the sign of $\det \mathbf{A}$.
- $\det(\mathbf{A}^T) = \det \mathbf{A}$.
- Multiplying any row or any column of $\mathbf{A}$ by a scalar k multiplies $\det \mathbf{A}$ by k.
- For any scalar k, $\det(k\mathbf{A}) = k^n \det \mathbf{A}$.
- Adding a multiple of one row of $\mathbf{A}$ to another row does not change $\det \mathbf{A}$. Nor does the corresponding operation for a pair of columns.
- If any row or column of $\mathbf{A}$ consists entirely of zeros, then $\det \mathbf{A} = 0$.

A multiple can be negative or positive, so the last rule covers subtracting a multiple of one row from another row.

Exercise 41

Use the rules given above to show that the following determinant is zero:

$$\begin{vmatrix} 1 & 3 & -2 \\ -3 & 2 & 6 \\ -2 & 4 & 4 \end{vmatrix}.$$

(*Hint*: Try to make a row or column vanish.)

The application of determinants to vector products

The expression for a 3×3 determinant in equation (58) provides a useful mnemonic for the definition of a vector product, given in Subsection 2.2. In fact, the vector product of two vectors can be obtained as follows.

Vector product as a determinant

If $\mathbf{b} = b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k}$ and $\mathbf{c} = c_x\mathbf{i} + c_y\mathbf{j} + c_z\mathbf{k}$ are two vectors, then their vector product can be obtained by evaluating the determinant

$$\mathbf{b} \times \mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} = \begin{vmatrix} b_y & b_z \\ c_y & c_z \end{vmatrix}\mathbf{i} - \begin{vmatrix} b_x & b_z \\ c_x & c_z \end{vmatrix}\mathbf{j} + \begin{vmatrix} b_x & b_y \\ c_x & c_y \end{vmatrix}\mathbf{k}. \qquad (61)$$

This reproduces the vector product rule that was given in equation (32) (though there we called the vectors $\mathbf{a}$ and $\mathbf{b}$).

In a similar way, the component expression for the scalar triple product of three vectors can be obtained as follows.

Scalar triple product as a determinant

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}$$
$$= a_x \begin{vmatrix} b_y & b_z \\ c_y & c_z \end{vmatrix} - a_y \begin{vmatrix} b_x & b_z \\ c_x & c_z \end{vmatrix} + a_z \begin{vmatrix} b_x & b_y \\ c_x & c_y \end{vmatrix}. \qquad (62)$$

Using these determinant-based expressions, it is easy to find similar formulas for the areas of parallelograms and the volumes of parallelepipeds. Many other geometric results can also be expressed in terms of determinants.

Exercise 42

Use determinant-based methods to do the following.

(a) Evaluate $\mathbf{r}\times\mathbf{s}$, where $\mathbf{r} = (2, 5, 0)$ and $\mathbf{s} = (2, -1, 2)$.

(b) Find the area of the parallelogram with sides defined by the position vectors $\mathbf{r}_1 = (1, 1, 0)$ and $\mathbf{r}_2 = (2, 2, -2)$.

(c) Find the volume of the parallelepiped with sides defined by $\mathbf{a} = (2, 5, 0)$, $\mathbf{b} = (1, -2, 0)$ and $\mathbf{c} = (1, -3, 2)$.

Finding inverse matrices

Determinants are very common in the physical sciences. They are often used as a neat way of summarising important results, as you will see shortly. However, for our immediate purposes, in the context of matrix algebra, determinants are important for the part they play in relation to inverse matrices. Here is a procedure for finding the inverse of any square invertible matrix, based on a generalisation of the method for 2×2 matrices in Section 3. (It is worth noting that for large matrices there are other methods that are computationally more efficient, and more likely to be used in practice.)

Procedure 1 Finding the inverse of an $n\times n$ matrix

Suppose that we are given the $n\times n$ matrix $\mathbf{A} = [a_{ij}]$.

1. Evaluate $\det\mathbf{A}$, and confirm that $\det\mathbf{A} \neq 0$. (If $\det\mathbf{A} = 0$, the matrix $\mathbf{A}$ is **non-invertible**; no inverse exists, and you should abandon the attempt to find it.)

2. Evaluate the cofactor C_{ij} of each element a_{ij}, using the relation $C_{ij} = (-1)^{i+j} M_{ij}$, where M_{ij} is the minor obtained by deleting row i and column j of the original determinant and forming the determinant of what remains.
3. Form the $n \times n$ square matrix $\mathbf{C} = [C_{ij}]$, where the element of $\mathbf{C}$ in row i and column j is the cofactor C_{ij}.
4. Take the transpose of $\mathbf{C}$ to obtain the matrix $\mathbf{C}^T$.
5. Scale the matrix $\mathbf{C}^T$ by $1/\det \mathbf{A}$ to obtain the inverse of $\mathbf{A}$:

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \mathbf{C}^T. \tag{63}$$

Note that given a square matrix $\mathbf{A}$, the matrix $\mathbf{C}^T$ is sometimes called the **adjugate matrix** of $\mathbf{A}$, represented by $\operatorname{adj} \mathbf{A}$. For that reason you will sometimes see the inverse of $\mathbf{A}$ written (elsewhere) as $\mathbf{A}^{-1} = \operatorname{adj} \mathbf{A}/\det \mathbf{A}$.

Example 20

Use Procedure 1 to derive the inverse of a 2×2 matrix, given in equation (52).

Solution

For a general 2×2 matrix $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the minors are

$$M_{11} = d, \quad M_{12} = c, \quad M_{21} = b, \quad M_{22} = a.$$

Hence the matrix of the cofactors is

$$\mathbf{C} = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix},$$

and its transpose is

$$\mathbf{C}^T = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

The determinant of $\mathbf{A}$ is $\det \mathbf{A} = ad - bc$, so the inverse of $\mathbf{A}$ is

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \mathbf{C}^T = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$

in agreement with equation (52).

Example 21

In Example 18 we showed that the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & 2 & -1 \\ 3 & 5 & 1 \\ 1 & 2 & 1 \end{bmatrix}$$

has $\det \mathbf{A} = 1$, so it is an invertible matrix. Find $\mathbf{A}^{-1}$.

Solution

The cofactors of $\mathbf{A}$ are

$$C_{11} = +\begin{vmatrix} 5 & 1 \\ 2 & 1 \end{vmatrix} = 3, \quad C_{12} = -\begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} = -2, \quad C_{13} = +\begin{vmatrix} 3 & 5 \\ 1 & 2 \end{vmatrix} = 1,$$

$$C_{21} = -\begin{vmatrix} 2 & -1 \\ 2 & 1 \end{vmatrix} = -4, \quad C_{22} = +\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} = 3, \quad C_{23} = -\begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix} = -2,$$

$$C_{31} = +\begin{vmatrix} 2 & -1 \\ 5 & 1 \end{vmatrix} = 7, \quad C_{32} = -\begin{vmatrix} 2 & -1 \\ 3 & 1 \end{vmatrix} = -5, \quad C_{33} = +\begin{vmatrix} 2 & 2 \\ 3 & 5 \end{vmatrix} = 4.$$

Thus the matrix of cofactors is

$$\mathbf{C} = [C_{ij}] = \begin{bmatrix} 3 & -2 & 1 \\ -4 & 3 & -2 \\ 7 & -5 & 4 \end{bmatrix}.$$

Since we already know that in this case $\det \mathbf{A} = 1$, it follows from Procedure 1 that

$$\mathbf{A}^{-1} = \mathbf{C}^T = \begin{bmatrix} 3 & -4 & 7 \\ -2 & 3 & -5 \\ 1 & -2 & 4 \end{bmatrix}.$$

Though not required, it is always good practice to confirm the result $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$ by explicit matrix multiplication. In this case we get

$$\mathbf{A}\mathbf{A}^{-1} = \begin{bmatrix} 2 & 2 & -1 \\ 3 & 5 & 1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & -4 & 7 \\ -2 & 3 & -5 \\ 1 & -2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{I}.$$

Exercise 43

Find the inverse of the matrix $\mathbf{A}$ of Exercise 40(a), where we showed that $\det \mathbf{A} = 3$.

We should also note the following rules relating to matrices, inverses and determinants.

Rules for $n \times n$ matrices, inverses and determinants

- The matrix $\mathbf{A}$ can be inverted if and only if $\det \mathbf{A} \neq 0$, in which case $\det(\mathbf{A}^{-1}) = 1/\det \mathbf{A}$.
- For two $n \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$, we have $\det(\mathbf{AB}) = \det \mathbf{A} \, \det \mathbf{B}$.
- If $\det(\mathbf{AB}) \neq 0$, so $(\mathbf{AB})^{-1}$ exists, then $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.

Note the reversal of order in the last of these rules. This is very similar to the reversal that we saw when expressing the transpose of a product in terms of the product of the transposes.

Learning outcomes

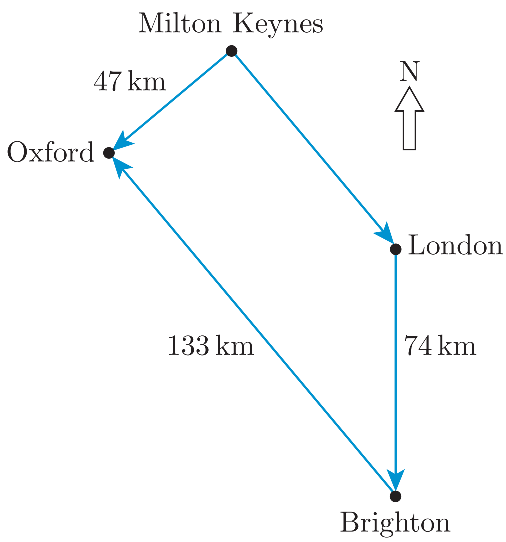
After studying this unit, you should be able to do the following.

- Understand the meaning of the terms scalar, vector, displacement vector, unit vector and position vector, and know what it means to say that two vectors are equal.
- Use vector notation and represent vectors as arrows on diagrams.
- Scale a vector by a scalar, and add two vectors geometrically using the triangle rule.
- Resolve a vector into its Cartesian components, and scale and add vectors given in Cartesian component form.
- Write down the vector equation of a straight line through two given points.
- Calculate the scalar product and vector product of two given vectors.
- Determine whether or not two given vectors are perpendicular or parallel to one another.
- Determine the magnitude of a vector and the angle between the directions of two vectors.
- Resolve a vector in a given direction.
- Use the vector product to determine the area of a parallelogram and the volume of a parallelepiped.
- Understand that a matrix can be used to represent a linear transformation, and know what this means geometrically for a 2×2 matrix.
- Add, subtract and multiply matrices of suitable sizes, and multiply a matrix by a scalar.
- Understand the terms transpose of a matrix, zero matrix, identity matrix, inverse matrix, invertible matrix and non-invertible matrix.
- Evaluate the determinants and inverses of 2×2 and 3×3 matrices, and know how to perform such calculations for $n \times n$ matrices.
- Use the determinant of a matrix to evaluate vector products, areas and volumes.

Solutions to exercises

Solution to Exercise 1

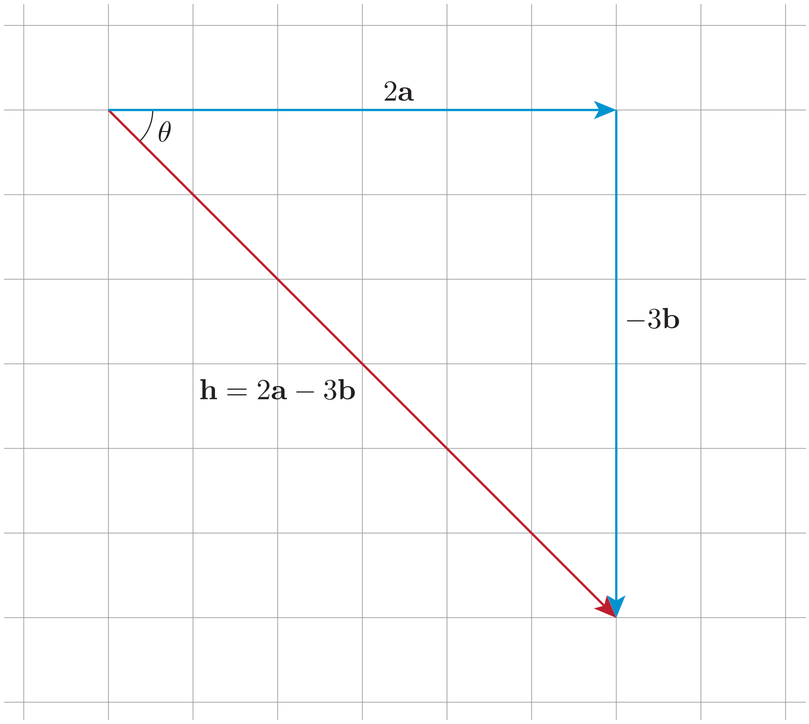
A sketch map is shown below.



From the map, the approximate distance between Milton Keynes and London is 80 km, and the displacement from Milton Keynes to London is about 80 km South-East. The distance between Milton Keynes and London can be described as the *magnitude* of the displacement from Milton Keynes to London.

Solution to Exercise 2

(a) The equation $\mathbf{h} = 2\mathbf{a} - 3\mathbf{b}$ is interpreted as $\mathbf{h} = 2\mathbf{a} + 3(-\mathbf{b})$, so $\mathbf{h}$ can be represented by the red arrow shown in the figure below.



The arrow for $\mathbf{a}$ is 3 units long, and the arrow for $\mathbf{b}$ is 2 units long. So the arrow for $2\mathbf{a}$ is 6 units long, and the arrow for $-3\mathbf{b}$ is 6 units long. These two arrows are perpendicular. The arrow for $\mathbf{h}$ forms the hypotenuse of a right-angled triangle, so we have

$$|\mathbf{h}| = \sqrt{6^2 + 6^2} = 6\sqrt{2}.$$

(b) Let θ be the angle between the directions of $\mathbf{h}$ and $2\mathbf{a}$. Then the diagram shows that $\tan\theta = 6/6 = 1$, so $\theta = \pi/4$ radians.

Solution to Exercise 3

The positive y-axis will be on your left.

Solution to Exercise 4

Systems (b), (c) and (d) are right-handed. System (a) is left-handed.

Solution to Exercise 5

(a) By visual inspection, the vectors are

$$\mathbf{a} = 2\mathbf{i} + \mathbf{j} = (2, 1) \quad \text{and} \quad \mathbf{b} = -2\mathbf{i} - 3\mathbf{j} = (-2, -3).$$

(b) Using equation (4), the magnitudes are

$$a = \sqrt{2^2 + 1^2} = \sqrt{5} \quad \text{and} \quad b = \sqrt{(-2)^2 + (-3)^2} = \sqrt{13}.$$

Using equation (5), the angle between $\mathbf{a}$ and the positive x-direction is given by

$$\cos\theta_x = a_x/a = 2/\sqrt{5} = 0.8944 \quad \text{(to 4 d.p.)},$$

so

$$\theta_x = \arccos(0.8944) = 0.464 \text{ radians}.$$

For $\mathbf{b}$, equation (5) gives

$$\cos\theta_x = b_x/b = -2/\sqrt{13} = -0.5547 \quad \text{(to 4 d.p.)},$$

so

$$\theta_x = \arccos(-0.5547) = 2.159 \text{ radians}.$$

Solution to Exercise 6

(a) $|\mathbf{a}| = \sqrt{2^2 + (-1)^2} = \sqrt{5}$,

$|\mathbf{b}| = \sqrt{1^2 + 3^2 + 5^2} = \sqrt{35}$.

(b) We have

$$\theta_x = \arccos\left(\frac{a_x}{a}\right) = \arccos\left(\frac{2}{\sqrt{5}}\right) = 0.4636 \text{ radians},$$

$$\theta_y = \arccos\left(\frac{a_y}{a}\right) = \arccos\left(\frac{-1}{\sqrt{5}}\right) = 2.0344 \text{ radians},$$

$$\theta_z = \arccos\left(\frac{a_z}{a}\right) = \arccos\left(\frac{0}{\sqrt{5}}\right) = \pi/2 \text{ radians}.$$

So $\mathbf{a}$ lies in the xy-plane, between the positive x-axis and the negative y-axis.

(c) $\mathbf{a} + \mathbf{b} = 3\mathbf{i} + 2\mathbf{j} + 5\mathbf{k}$,

$2\mathbf{a} - \mathbf{b} = 3\mathbf{i} - 5\mathbf{j} - 5\mathbf{k}$,

$\mathbf{c} + 2\mathbf{b} - 3\mathbf{a} = -4\mathbf{i} + 10\mathbf{j} + 8\mathbf{k}$.

(d) We have $\overrightarrow{PQ} = 2\mathbf{a} - \mathbf{b} = 3\mathbf{i} - 5\mathbf{j} - 5\mathbf{k}$. The point Q has the position vector $\overrightarrow{OQ}$, which is given by

$$\begin{aligned}\overrightarrow{OQ} &= \overrightarrow{OP} + \overrightarrow{PQ}\\ &= (2\mathbf{j} + 3\mathbf{k}) + (3\mathbf{i} - 5\mathbf{j} - 5\mathbf{k})\\ &= 3\mathbf{i} - 3\mathbf{j} - 2\mathbf{k},\end{aligned}$$

so Q is the point $(3, -3, -2)$.

(e) We have $\overrightarrow{RS} = \mathbf{a} + 2\mathbf{b} = 4\mathbf{i} + 5\mathbf{j} + 10\mathbf{k}$. The point S has the position vector $\overrightarrow{OS}$, which is given by

$$\begin{aligned}\overrightarrow{OS} &= \overrightarrow{OR} + \overrightarrow{RS} \\ &= (\mathbf{i} + \mathbf{j}) + (4\mathbf{i} + 5\mathbf{j} + 10\mathbf{k}) \\ &= 5\mathbf{i} + 6\mathbf{j} + 10\mathbf{k},\end{aligned}$$

so S is the point $(5, 6, 10)$.

Solution to Exercise 7

The magnitude of any vector is given by the positive square root of the sum of the squares of its components. In the case of the unit vector $\widehat{\mathbf{a}}$, it follows that the magnitude is

$$\widehat{a} = \sqrt{\left(\frac{a_x}{a}\right)^2 + \left(\frac{a_y}{a}\right)^2 + \left(\frac{a_z}{a}\right)^2},$$

that is,

$$\widehat{a} = \frac{\sqrt{a_x^2 + a_y^2 + a_z^2}}{a} = \frac{a}{a} = 1,$$

as required.

Solution to Exercise 8

Relative to the origin of the Cartesian coordinate system, the two points have position vectors $\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and $2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$. Thus the vector equation of the line is

$$\begin{aligned}\mathbf{r}(t) &= (1 - t)(\mathbf{i} + \mathbf{j} + 2\mathbf{k}) + t(2\mathbf{i} + 3\mathbf{j} + \mathbf{k}) \\ &= (1 + t)\mathbf{i} + (1 + 2t)\mathbf{j} + (2 - t)\mathbf{k},\end{aligned}$$

where $-\infty < t < \infty$.

Solution to Exercise 9

The acceleration is given by

$$\mathbf{a}(t) = \dot{\mathbf{v}}(t) = \frac{d}{dt}(6t, 4t^3, -1) = (6, 12t^2, 0).$$

Solution to Exercise 10

$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos\theta = 2 \times 4 \times \cos\frac{\pi}{3} = 4,$

$\mathbf{b} \cdot \mathbf{c} = |\mathbf{b}|\,|\mathbf{c}| \cos\theta = 4 \times 1 \times \cos\frac{\pi}{6} = 2\sqrt{3},$

$\mathbf{a} \cdot \mathbf{c} = |\mathbf{a}|\,|\mathbf{c}| \cos\theta = 2 \times 1 \times \cos\left(\frac{\pi}{3} + \frac{\pi}{6}\right) = 2\cos\frac{\pi}{2} = 0,$

$\mathbf{b} \cdot \mathbf{b} = |\mathbf{b}|\,|\mathbf{b}| \cos\theta = 4 \times 4 \times \cos 0 = 16.$

Solution to Exercise 11

(a)
$$\begin{aligned}(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}-\mathbf{b}) &= \mathbf{a}\cdot(\mathbf{a}-\mathbf{b})+\mathbf{b}\cdot(\mathbf{a}-\mathbf{b})\\ &= \mathbf{a}\cdot\mathbf{a}-\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{a}-\mathbf{b}\cdot\mathbf{b}\\ &= \mathbf{a}\cdot\mathbf{a}-\mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{b}-\mathbf{b}\cdot\mathbf{b}\\ &= \mathbf{a}\cdot\mathbf{a}-\mathbf{b}\cdot\mathbf{b}.\end{aligned}$$

(b) From equation (17),
$$\begin{aligned}|\mathbf{a}+\mathbf{b}|^2 &= (\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b})\\ &= \mathbf{a}\cdot(\mathbf{a}+\mathbf{b})+\mathbf{b}\cdot(\mathbf{a}+\mathbf{b})\\ &= \mathbf{a}\cdot\mathbf{a}+\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{a}+\mathbf{b}\cdot\mathbf{b}\\ &= \mathbf{a}\cdot\mathbf{a}+\mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{b}\\ &= \mathbf{a}\cdot\mathbf{a}+2\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{b}.\end{aligned}$$

(c) When $\mathbf{a}$ and $\mathbf{b}$ are antiparallel, the angle between them is $\theta = \pi$, consequently $\cos\theta = -1$ and $\mathbf{a}\cdot\mathbf{b} = -|\mathbf{a}|\,|\mathbf{b}| = -ab$.

Solution to Exercise 12

(a) $\mathbf{a}\cdot\mathbf{b} = (4\times 1)+(1\times -3)+(-5\times 1) = -4$.

The negative sign tells us that the angle between $\mathbf{a}$ and $\mathbf{b}$ is between $\frac{\pi}{2}$ and π radians, i.e. it is an obtuse angle.

(b) No, the scalar product is $\mathbf{c}\cdot\mathbf{d} = 9-5+4 = 8$. Since this is not equal to zero, the pair of vectors fails the test for orthogonality.

Solution to Exercise 13

For orthogonality, we require $(\mathbf{p}+\lambda\mathbf{q})\cdot\mathbf{r} = 0$. This condition gives
$$(\mathbf{p}+\lambda\mathbf{q})\cdot\mathbf{r} = (\mathbf{p}\cdot\mathbf{r})+\lambda(\mathbf{q}\cdot\mathbf{r}) = 0.$$
We have
$$\mathbf{p}\cdot\mathbf{r} = (3\mathbf{i}+2\mathbf{j}-\mathbf{k})\cdot(2\mathbf{i}-\mathbf{j}-\mathbf{k}) = 6-2+1 = 5$$
and
$$\mathbf{q}\cdot\mathbf{r} = (-\mathbf{i}+\mathbf{j}+2\mathbf{k})\cdot(2\mathbf{i}-\mathbf{j}-\mathbf{k}) = -2-1-2 = -5,$$
so $(\mathbf{p}+\lambda\mathbf{q})\cdot\mathbf{r} = 5-5\lambda = 0$, hence $\lambda = 1$.

Solution to Exercise 14

(a) $\mathbf{a}\cdot\mathbf{c} = -2-3+3 = -2$, $\mathbf{a}\cdot\mathbf{d} = -4+0+1 = -3$ and $\mathbf{a}\cdot\mathbf{e} = -2+3-1 = 0$. Thus only $\mathbf{e}$ is perpendicular to $\mathbf{a}$.

(b) We have
$$\mathbf{a}+2\mathbf{b} = \mathbf{j}+9\mathbf{k}.$$
The displacement from the origin to the point $(1,1,1)$ is represented by the position vector $\mathbf{r} = \mathbf{i}+\mathbf{j}+\mathbf{k}$. The corresponding unit vector is $\widehat{\mathbf{r}} = \frac{1}{\sqrt{3}}(\mathbf{i}+\mathbf{j}+\mathbf{k})$. The component of $\mathbf{a}+2\mathbf{b}$ in the direction of the specified displacement is therefore
$$\widehat{\mathbf{r}}\cdot(\mathbf{a}+2\mathbf{b}) = \tfrac{1}{\sqrt{3}}(\mathbf{i}+\mathbf{j}+\mathbf{k})\cdot(\mathbf{j}+9\mathbf{k}) = \tfrac{10}{\sqrt{3}}.$$

(c) First we need to find a unit vector in the direction of the vector $\mathbf{a} - 2\mathbf{b}$. Working in ordered triple notation, $\mathbf{a} - 2\mathbf{b} = (4, -7, -7)$ and its magnitude is $|\mathbf{a} - 2\mathbf{b}| = \sqrt{16 + 49 + 49} = \sqrt{114}$. Consequently, a unit vector in the direction of $\mathbf{a} - 2\mathbf{b}$ is given by $(\mathbf{a} - 2\mathbf{b})/\sqrt{114} = (4, -7, -7)/\sqrt{114}$. So the required component is

$$\begin{aligned}(\mathbf{a} + 2\mathbf{b}) \cdot \tfrac{1}{\sqrt{114}}(\mathbf{a} - 2\mathbf{b}) &= (0, 1, 9) \cdot \tfrac{1}{\sqrt{114}}(4, -7, -7) \\ &= \tfrac{-70}{\sqrt{114}} \\ &= -6.56 \quad \text{(to 2 d.p.)}.\end{aligned}$$

Solution to Exercise 15

Since $\mathbf{v}$ has magnitude 4, and makes an angle of $2\pi/3$ with the *positive* x-axis, its component along the $\mathbf{i}$-direction is

$$\mathbf{v} \cdot \mathbf{i} = |\mathbf{v}|\,|\mathbf{i}| \cos(2\pi/3) = -\frac{4}{2} = -2.$$

Since $\mathbf{v}$ makes an angle of $\pi/6$ with the *positive* y-axis, its component along the $\mathbf{j}$-direction is

$$\mathbf{v} \cdot \mathbf{j} = |\mathbf{v}|\,|\mathbf{j}| \cos(\pi/6) = 4\frac{\sqrt{3}}{2} = 2\sqrt{3}.$$

Hence we can write $\mathbf{v}$ as

$$\mathbf{v} = -2\mathbf{i} + 2\sqrt{3}\,\mathbf{j}.$$

Solution to Exercise 16

Since $\widehat{\mathbf{u}}$, $\widehat{\mathbf{v}}$ and $\widehat{\mathbf{w}}$ are mutually perpendicular unit vectors, we can write

$$\mathbf{a} = a_u\widehat{\mathbf{u}} + a_v\widehat{\mathbf{v}} + a_w\widehat{\mathbf{w}},$$

where a_u, a_v and a_w are the components of $\mathbf{a}$ along the $\widehat{\mathbf{u}}$, $\widehat{\mathbf{v}}$ and $\widehat{\mathbf{w}}$ directions, respectively. Hence

$$\begin{aligned}a_u &= \mathbf{a} \cdot \widehat{\mathbf{u}} = \frac{1}{\sqrt{2}}(2, 1, 0) \cdot (1, 0, 1) = \frac{2}{\sqrt{2}} = \sqrt{2}, \\ a_v &= \mathbf{a} \cdot \widehat{\mathbf{v}} = \frac{1}{\sqrt{2}}(2, 1, 0) \cdot (1, 0, -1) = \frac{2}{\sqrt{2}} = \sqrt{2}, \\ a_w &= \mathbf{a} \cdot \widehat{\mathbf{w}} = (2, 1, 0) \cdot (0, 1, 0) = 1.\end{aligned}$$

So

$$\mathbf{a} = \sqrt{2}\,\widehat{\mathbf{u}} + \sqrt{2}\,\widehat{\mathbf{v}} + \widehat{\mathbf{w}}.$$

We can check that this solution is correct by substituting in the numerical values for $\widehat{\mathbf{u}}$, $\widehat{\mathbf{v}}$ and $\widehat{\mathbf{w}}$:

$$\mathbf{a} = \sqrt{2}\frac{1}{\sqrt{2}}(1, 0, 1) + \sqrt{2}\frac{1}{\sqrt{2}}(1, 0, -1) + (0, 1, 0) = (2, 1, 0),$$

as required.

Solution to Exercise 17

(a) Since $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$ for any vectors $\mathbf{a}$ and $\mathbf{b}$, we have

$$\mathbf{j} \times \mathbf{i} = -\mathbf{k}, \quad \mathbf{k} \times \mathbf{j} = -\mathbf{i} \quad \text{and} \quad \mathbf{i} \times \mathbf{k} = -\mathbf{j}.$$

(b) By definition, $\mathbf{a} \times \mathbf{a} = \mathbf{0}$ for any vector $\mathbf{a}$, so we have

$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}.$$

(c)
$$\begin{aligned}
&(\mathbf{i} \times (\mathbf{i} + \mathbf{k})) - ((\mathbf{i} + \mathbf{j}) \times \mathbf{k}) \\
&\quad = ((\mathbf{i} \times \mathbf{i}) + (\mathbf{i} \times \mathbf{k})) - ((\mathbf{i} \times \mathbf{k}) + (\mathbf{j} \times \mathbf{k})) \\
&\quad = (\mathbf{0} + (-\mathbf{j})) - (-\mathbf{j} + \mathbf{i}) \\
&\quad = -\mathbf{i}.
\end{aligned}$$

(d)
$$\begin{aligned}
&(\mathbf{i} + \mathbf{k}) \times (\mathbf{i} + \mathbf{j} + \mathbf{k}) \\
&\quad = (\mathbf{i} \times (\mathbf{i} + \mathbf{j} + \mathbf{k})) + (\mathbf{k} \times (\mathbf{i} + \mathbf{j} + \mathbf{k})) \\
&\quad = (\mathbf{0} + \mathbf{k} + (-\mathbf{j})) + (\mathbf{j} + (-\mathbf{i}) + \mathbf{0}) \\
&\quad = -\mathbf{i} + \mathbf{k}.
\end{aligned}$$

Solution to Exercise 18

Since the origin is one of the three points, the sides of the triangle will be defined by the position vectors of the other two points, that is, by the vectors $\mathbf{r}_1 = (2, 1, 1)$ and $\mathbf{r}_2 = (1, -1, -1)$. The area of the triangle will be half the area of the parallelogram with sides defined by the vectors $\mathbf{r}_1$ and $\mathbf{r}_2$. It therefore follows from the expression for the area of a parallelogram that the area of the triangle is

$$\begin{aligned}
\tfrac{1}{2}|\mathbf{r}_1 \times \mathbf{r}_2| &= \tfrac{1}{2}|(2, 1, 1) \times (1, -1, -1)| \\
&= \tfrac{1}{2}|(1(-1) - 1(-1), 1(1) - 2(-1), 2(-1) - 1(1))| \\
&= \tfrac{1}{2}|(0, 3, -3)| \\
&= \tfrac{1}{2}\sqrt{3^2 + (-3)^2} = \tfrac{1}{2}\sqrt{18} = \frac{3}{\sqrt{2}}.
\end{aligned}$$

Solution to Exercise 19

Since the origin is one of the corners, two adjoining sides of the parallelogram will be defined by the position vectors $\mathbf{r}_1 = (a, b, 0)$ and $\mathbf{r}_2 = (c, d, 0)$. Using the vector product method, the area of this parallelogram is

$$\begin{aligned}
|\mathbf{r}_1 \times \mathbf{r}_2| &= |(a, b, 0) \times (c, d, 0)| \\
&= |(b(0) - 0(d), 0(c) - a(0), a(d) - b(c))| \\
&= |(0, 0, ad - bc)| \\
&= |ad - bc|.
\end{aligned}$$

If $b = c = 0$, then the corners are $(0, 0, 0)$, $(a, 0, 0)$, $(0, d, 0)$ and $(a, d, 0)$, so the parallelogram is a rectangle. The above formula for the area becomes $|ad|$, which is as expected.

Solution to Exercise 20

Given that both $\mathbf{a} \times \mathbf{b}$ and $\mathbf{c}$ are non-zero, $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \mathbf{0}$ implies that the angle between the direction of $\mathbf{a} \times \mathbf{b}$ and the direction of $\mathbf{c}$ must be 0 or π radians (so that its sine is zero). However, $\mathbf{a} \times \mathbf{b}$ is always perpendicular to the plane containing $\mathbf{a}$ and $\mathbf{b}$, so we can say that $\mathbf{c}$, which is parallel or antiparallel to $\mathbf{a} \times \mathbf{b}$, must also be perpendicular to the plane containing $\mathbf{a}$ and $\mathbf{b}$.

Solution to Exercise 21

(a) is 2×2 – a square matrix.

(b) is 3×3 – a square matrix.

(c) is 1×2 – a row matrix.

(d) is 3×1 – a column matrix.

Solution to Exercise 22

(a) $\mathbf{A}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 13 \end{bmatrix}$.

(b) $\mathbf{A}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(c) $\mathbf{A}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Solution to Exercise 23

(a) Either by examining the columns of $\mathbf{D}$, or by multiplying $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ by $\mathbf{D}$, we see that $(1, 0)$ is mapped to $(3, 0)$, and $(0, 1)$ is mapped to $(0, 2)$.

(b)
$$\mathbf{D}\begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 9 \\ -4 \end{bmatrix},$$
so $(3, -2)$ is mapped to $(9, -4)$.

(c) The transformation rescales the sides of a unit square in proportion to the rescaling of the unit vectors. Since $(1, 0)$ is mapped to $(3, 0)$, and $(0, 1)$ is mapped to $(0, 2)$, the area of the unit square will be enlarged from $1 \times 1 = 1$ to $3 \times 2 = 6$.

Solution to Exercise 24

(a) With $\alpha = \frac{\pi}{4}$, the required rotation matrix is
$$\mathbf{R}(\tfrac{\pi}{4}) = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

(b) From the columns of $\mathbf{R}(\frac{\pi}{4})$ (or by using matrix multiplication) it can be seen that $(1, 0)$ is mapped to $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, and $(0, 1)$ is mapped to $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

(c) $$\mathbf{R}(\tfrac{\pi}{4})\begin{bmatrix}-1\\-1\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}-1\\-1\end{bmatrix} = \begin{bmatrix}0\\-\frac{2}{\sqrt{2}}\end{bmatrix} = \begin{bmatrix}0\\-\sqrt{2}\end{bmatrix},$$

so $(-1,-1)$ is mapped to $(0,-\sqrt{2})$.

(d) $(1,0)$ is mapped to $\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$, and $(0,1)$ is mapped to $\left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$. However, $\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$ and $\left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$ are also unit vectors. So the unit vectors are mapped to unit vectors, and a unit square remains a unit square. So the area of a unit square is not changed by the transformation. (This is actually a general feature of rotations, not restricted to the case $\alpha = \frac{\pi}{4}$.)

Solution to Exercise 25

$\mathbf{D}(\kappa,\lambda) = \mathbf{I}$ when $\kappa = 1$ and $\lambda = 1$.

$\mathbf{R}(\alpha) = \mathbf{I}$ when α is zero or any even multiple of π, i.e. $\alpha = 2n\pi$, where n is any integer.

Solution to Exercise 26

(a) $3\begin{bmatrix}1\\0\end{bmatrix} - 2\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}3\\-2\end{bmatrix}$.

(b) $4\left(\begin{bmatrix}1\\1\end{bmatrix} - 3\begin{bmatrix}1\\-1\end{bmatrix}\right) = \begin{bmatrix}-8\\16\end{bmatrix}$.

(c) $2\begin{bmatrix}1 & 0\\-2 & 3\end{bmatrix} = \begin{bmatrix}2 & 0\\-4 & 6\end{bmatrix}$.

(d) $2\left(\begin{bmatrix}1 & 0\\-2 & 3\end{bmatrix} + 2\begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}\right) = \begin{bmatrix}2 & 4\\-8 & 6\end{bmatrix}$.

(e) $2\left(\begin{bmatrix}a & b\\c & d\end{bmatrix} + 2\begin{bmatrix}a & 2b\\-2c & d\end{bmatrix}\right) = \begin{bmatrix}6a & 10b\\-6c & 6d\end{bmatrix}$.

Solution to Exercise 27

(a) $$\begin{bmatrix}2 & 0\\1 & 1\end{bmatrix}\begin{bmatrix}0 & 3\\1 & 1\end{bmatrix} = \begin{bmatrix}2(0)+0(1) & 2(3)+0(1)\\1(0)+1(1) & 1(3)+1(1)\end{bmatrix} = \begin{bmatrix}0 & 6\\1 & 4\end{bmatrix}.$$

(b) $$\begin{bmatrix}1 & 2\\2 & -3\end{bmatrix}\begin{bmatrix}1 & 3\\2 & -1\end{bmatrix} = \begin{bmatrix}1(1)+2(2) & 1(3)+2(-1)\\2(1)-3(2) & 2(3)-3(-1)\end{bmatrix} = \begin{bmatrix}5 & 1\\-4 & 9\end{bmatrix}.$$

(c) $$\begin{bmatrix}1 & 3\\2 & -1\end{bmatrix}\begin{bmatrix}1 & 2\\2 & -3\end{bmatrix} = \begin{bmatrix}1(1)+3(2) & 1(2)+3(-3)\\2(1)-1(2) & 2(2)-1(-3)\end{bmatrix} = \begin{bmatrix}7 & -7\\0 & 7\end{bmatrix}.$$

The solutions to parts (b) and (c) show that the two matrices do not commute.

Solution to Exercise 28

The required matrix is given by the matrix product $\mathbf{C} = \mathbf{R}\left(\frac{\pi}{4}\right)\mathbf{D}(2,1)$. Recalling that $\sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$, we have

$$\begin{aligned}\mathbf{C} = \begin{bmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix} &= \begin{bmatrix}\frac{1}{\sqrt{2}}(2) - \frac{1}{\sqrt{2}}(0) & \frac{1}{\sqrt{2}}(0) - \frac{1}{\sqrt{2}}(1)\\ \frac{1}{\sqrt{2}}(2) + \frac{1}{\sqrt{2}}(0) & \frac{1}{\sqrt{2}}(0) + \frac{1}{\sqrt{2}}(1)\end{bmatrix}\\ &= \begin{bmatrix}2\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\ 2\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}.\end{aligned}$$

Solution to Exercise 29

$$\begin{aligned}\mathbf{F} = \begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix} &= \begin{bmatrix}2(\frac{1}{\sqrt{2}}) + 0(\frac{1}{\sqrt{2}}) & 2(-\frac{1}{\sqrt{2}}) + 0(\frac{1}{\sqrt{2}})\\ 0(\frac{1}{\sqrt{2}}) + 1(\frac{1}{\sqrt{2}}) & 0(-\frac{1}{\sqrt{2}}) + 1(\frac{1}{\sqrt{2}})\end{bmatrix}\\ &= \begin{bmatrix}2\frac{1}{\sqrt{2}} & -2\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}.\end{aligned}$$

$\mathbf{F}$ clearly differs from $\mathbf{C}$.

Solution to Exercise 30

We have

$$\begin{aligned}\mathbf{R}(\alpha)\,\mathbf{R}(\beta) &= \begin{bmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{bmatrix}\begin{bmatrix}\cos\beta & -\sin\beta\\ \sin\beta & \cos\beta\end{bmatrix}\\ &= \begin{bmatrix}\cos\alpha\cos\beta - \sin\alpha\sin\beta & -\cos\alpha\sin\beta - \sin\alpha\cos\beta\\ \sin\alpha\cos\beta + \cos\alpha\sin\beta & -\sin\alpha\sin\beta + \cos\alpha\cos\beta\end{bmatrix}.\end{aligned}$$

Now use the trigonometric identities

$$\begin{aligned}&\cos\alpha\cos\beta \pm \sin\alpha\sin\beta = \cos(\alpha \mp \beta),\\ &\sin\alpha\cos\beta \pm \cos\alpha\sin\beta = \sin(\alpha \pm \beta)\end{aligned}$$

to get

$$\mathbf{R}(\alpha)\,\mathbf{R}(\beta) = \begin{bmatrix}\cos(\alpha+\beta) & -\sin(\alpha+\beta)\\ \sin(\alpha+\beta) & \cos(\alpha+\beta)\end{bmatrix}.$$

But the right-hand side of this equation is just the matrix $\mathbf{R}(\alpha+\beta)$.

Hence we have shown that

$$\mathbf{R}(\alpha)\,\mathbf{R}(\beta) = \mathbf{R}(\alpha+\beta).$$

Now we can either calculate $\mathbf{R}(\beta)\,\mathbf{R}(\alpha)$, and show explicitly that $\mathbf{R}(\alpha)\,\mathbf{R}(\beta) = \mathbf{R}(\beta)\,\mathbf{R}(\alpha)$, or we can save ourselves some work by noticing that

$$\mathbf{R}(\alpha)\,\mathbf{R}(\beta) = \mathbf{R}(\alpha+\beta) = \mathbf{R}(\beta+\alpha) = \mathbf{R}(\beta)\,\mathbf{R}(\alpha).$$

So $\mathbf{R}(\alpha)$ and $\mathbf{R}(\beta)$ commute, as expected on geometric grounds.

Solution to Exercise 31

Remembering that $\cos(-\alpha) = \cos(\alpha)$ and $\sin(-\alpha) = -\sin(\alpha)$, we have

$$\begin{aligned}
\mathbf{R}(\alpha)\,\mathbf{R}(-\alpha) &= \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos(-\alpha) & -\sin(-\alpha) \\ \sin(-\alpha) & \cos(-\alpha) \end{bmatrix} \\
&= \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \\
&= \begin{bmatrix} \cos^2\alpha + \sin^2\alpha & 0 \\ 0 & \cos^2\alpha + \sin^2\alpha \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathbf{R}(-\alpha)\,\mathbf{R}(\alpha) &= \begin{bmatrix} \cos(-\alpha) & -\sin(-\alpha) \\ \sin(-\alpha) & \cos(-\alpha) \end{bmatrix} \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \\
&= \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \\
&= \begin{bmatrix} \cos^2\alpha + \sin^2\alpha & 0 \\ 0 & \cos^2\alpha + \sin^2\alpha \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}.
\end{aligned}$$

So $\mathbf{R}(\alpha)$ is indeed the inverse of $\mathbf{R}(-\alpha)$, as claimed.

In fact, there was an easier way to prove this. From Exercise 30, we know that $\mathbf{R}(\alpha)\,\mathbf{R}(\beta) = \mathbf{R}(\beta)\,\mathbf{R}(\alpha) = \mathbf{R}(\alpha+\beta)$. Furthermore from Exercise 25, we know that $\mathbf{R}(0) = \mathbf{I}$. Hence

$$\mathbf{R}(\alpha)\,\mathbf{R}(-\alpha) = \mathbf{R}(-\alpha)\,\mathbf{R}(\alpha) = \mathbf{R}(\alpha-\alpha) = \mathbf{R}(0) = \mathbf{I}.$$

Solution to Exercise 32

$$\begin{aligned}
\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} &= \frac{1}{ad-bc}\begin{bmatrix} ad-bc & -ab+ba \\ cd-dc & -cb+da \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
\end{aligned}$$

Solution to Exercise 33

(a) Using the general inversion formula,

$$\mathbf{A}^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix},$$

$$\mathbf{B}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

$$\mathbf{C}^{-1} = \tfrac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

For the matrix $\mathbf{D} = \mathbf{ABC}$, the product of three matrices can be interpreted as $(\mathbf{AB})\mathbf{C}$ or $\mathbf{A}(\mathbf{BC})$ since matrix multiplication is associative. Adopting the first option,

$$\begin{aligned}\mathbf{D} = \mathbf{ABC} = (\mathbf{AB})\mathbf{C} &= \left(\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\right)\begin{bmatrix}1 & -1\\1 & 1\end{bmatrix}\\ &= \begin{bmatrix}1 & -1\\0 & -1\end{bmatrix}\begin{bmatrix}1 & -1\\1 & 1\end{bmatrix}\\ &= \begin{bmatrix}0 & -2\\-1 & -1\end{bmatrix}.\end{aligned}$$

Applying the matrix inversion formula,

$$\mathbf{D}^{-1} = (\mathbf{ABC})^{-1} = -\tfrac{1}{2}\begin{bmatrix}-1 & 2\\1 & 0\end{bmatrix} = \tfrac{1}{2}\begin{bmatrix}1 & -2\\-1 & 0\end{bmatrix}.$$

(b) We have

$$\mathbf{B}^{-1}\mathbf{A}^{-1} = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}1 & -1\\0 & 1\end{bmatrix} = \begin{bmatrix}1 & -1\\0 & -1\end{bmatrix},$$

hence

$$\mathbf{C}^{-1}\mathbf{B}^{-1}\mathbf{A}^{-1} = \tfrac{1}{2}\begin{bmatrix}1 & 1\\-1 & 1\end{bmatrix}\begin{bmatrix}1 & -1\\0 & -1\end{bmatrix} = \tfrac{1}{2}\begin{bmatrix}1 & -2\\-1 & 0\end{bmatrix},$$

which is indeed equal to $\mathbf{D}^{-1} = (\mathbf{ABC})^{-1}$.

Solution to Exercise 34

$\begin{bmatrix}-2 & 4\\-1 & 2\end{bmatrix}$ has no inverse, because its determinant is $-2 \times 2 - (-1 \times 4) = 0$.

In the other three cases, the determinant is not zero, so inverses will exist.

Solution to Exercise 35

$\det \mathbf{A} = 1$, $\det \mathbf{B} = -1$ and $\det \mathbf{C} = 2$.

It was shown in Exercise 33 that

$$\mathbf{ABC} = \begin{bmatrix}0 & -2\\-1 & -1\end{bmatrix}.$$

Consequently, $\det(\mathbf{ABC}) = -2$.

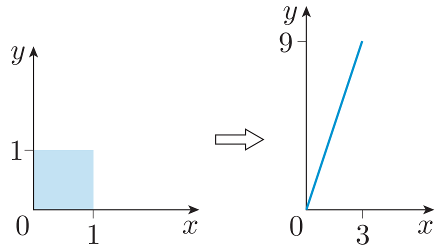
Thus $\det(\mathbf{ABC}) = \det \mathbf{A} \det \mathbf{B} \det \mathbf{C}$.

You will see later that this is more generally true: the determinant of a product of matrices is always equal to the product of the individual determinants.

Solution to Exercise 36

The columns of $\mathbf{A}$ show the effect that it will have on the Cartesian unit (column) vectors $\mathbf{i}$ and $\mathbf{j}$: they will become the vectors $(1, 3)$ and $(2, 6)$, respectively. These vectors are parallel. Hence $\mathbf{A}$ does indeed transform the unit square to a geometric object with no area, namely to a finite portion of a straight line.

The point $(0,0)$ is transformed to the point $(0,0)$. The line in question is $y = 3x$, and the unit square is mapped to the portion with $0 \le x \le 3$, since the point with coordinates $(1,1)$ is mapped to the point $(3,9)$. The result is depicted in the figure below.



Solution to Exercise 37

(a) $\begin{bmatrix} 1 & 2 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 1(1)+2(2) & 1(3)+2(-1) \\ 2(1)-3(2) & 2(3)-3(-1) \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ -4 & 9 \end{bmatrix}.$

(b) $\begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 6 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 2(1)+1(0) & 2(6)+1(2) \end{bmatrix} = \begin{bmatrix} 2 & 14 \end{bmatrix}.$

(c) $\begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 3 & 0 & -4 \end{bmatrix} = \begin{bmatrix} 1(3) & 1(0) & 1(-4) \\ 2(3) & 2(0) & 2(-4) \end{bmatrix} = \begin{bmatrix} 3 & 0 & -4 \\ 6 & 0 & -8 \end{bmatrix}.$

(d)
$$\begin{bmatrix} 3 & 1 & 2 \\ 0 & 5 & 1 \end{bmatrix} \begin{bmatrix} -2 & 0 & 1 \\ 1 & 3 & 0 \\ 4 & 1 & -1 \end{bmatrix}$$
$$= \begin{bmatrix} 3(-2)+1(1)+2(4) & 3(0)+1(3)+2(1) & 3(1)+1(0)+2(-1) \\ 0(-2)+5(1)+1(4) & 0(0)+5(3)+1(1) & 0(1)+5(0)+1(-1) \end{bmatrix}$$
$$= \begin{bmatrix} 3 & 5 & 1 \\ 9 & 16 & -1 \end{bmatrix}.$$

(e) The product does not exist, because the left-hand matrix is of order 3×1, whereas the right-hand matrix is of order 2×2.

Solution to Exercise 38

Taking care of the order specified,

$$\mathbf{C} = \mathbf{BA} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}$$

and

$$\mathbf{D} = \mathbf{AB} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

So $\mathbf{BA} \neq \mathbf{AB}$, i.e. the rotations $\mathbf{A}$ and $\mathbf{B}$ do not commute.

Solution to Exercise 39

(a) $\mathbf{A}^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$, $\mathbf{B}^T = \begin{bmatrix} 2 & -1 & 3 \\ 5 & -4 & 1 \end{bmatrix}$, $\mathbf{C}^T = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$.

(b) $$(\mathbf{A}+\mathbf{B})^T = \left(\begin{bmatrix}1 & 2\\ 3 & 4\\ 5 & 6\end{bmatrix} + \begin{bmatrix}2 & 5\\ -1 & -4\\ 3 & 1\end{bmatrix}\right)^T = \begin{bmatrix}3 & 7\\ 2 & 0\\ 8 & 7\end{bmatrix}^T = \begin{bmatrix}3 & 2 & 8\\ 7 & 0 & 7\end{bmatrix}$$

and

$$\mathbf{A}^T + \mathbf{B}^T = \begin{bmatrix}1 & 3 & 5\\ 2 & 4 & 6\end{bmatrix} + \begin{bmatrix}2 & -1 & 3\\ 5 & -4 & 1\end{bmatrix} = \begin{bmatrix}3 & 2 & 8\\ 7 & 0 & 7\end{bmatrix}.$$

Thus $(\mathbf{A}+\mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$.

(c) $$(\mathbf{AC})^T = \left(\begin{bmatrix}1 & 2\\ 3 & 4\\ 5 & 6\end{bmatrix}\begin{bmatrix}1 & 0\\ 2 & 3\end{bmatrix}\right)^T = \begin{bmatrix}5 & 6\\ 11 & 12\\ 17 & 18\end{bmatrix}^T = \begin{bmatrix}5 & 11 & 17\\ 6 & 12 & 18\end{bmatrix}$$

and

$$\mathbf{C}^T\mathbf{A}^T = \begin{bmatrix}1 & 2\\ 0 & 3\end{bmatrix}\begin{bmatrix}1 & 3 & 5\\ 2 & 4 & 6\end{bmatrix} = \begin{bmatrix}5 & 11 & 17\\ 6 & 12 & 18\end{bmatrix}.$$

Thus $(\mathbf{AC})^T = \mathbf{C}^T\mathbf{A}^T$.

Solution to Exercise 40

(a) The determinant of $\mathbf{A} = \begin{bmatrix}2 & -1 & 2\\ 5 & 1 & -1\\ 2 & 1 & -1\end{bmatrix}$ is

$$\begin{aligned}\det \mathbf{A} &= 2\begin{vmatrix}1 & -1\\ 1 & -1\end{vmatrix} + \begin{vmatrix}5 & -1\\ 2 & -1\end{vmatrix} + 2\begin{vmatrix}5 & 1\\ 2 & 1\end{vmatrix}\\ &= 2\times(-1+1) + (-5+2) + 2\times(5-2) = 0 - 3 + 6 = 3.\end{aligned}$$

(b) The determinant of $\mathbf{B} = \begin{bmatrix}0 & 2 & -1\\ 3 & 0 & -1\\ 2 & -1 & 0\end{bmatrix}$ is

$$\begin{aligned}\det \mathbf{B} &= 0 - 2\begin{vmatrix}3 & -1\\ 2 & 0\end{vmatrix} - 1\begin{vmatrix}3 & 0\\ 2 & -1\end{vmatrix}\\ &= -2\times(0+2) - 1\times(-3-0) = -4 + 3 = -1.\end{aligned}$$

Solution to Exercise 41

If we follow the last rule by adding twice the first column to the third, we do not change the determinant but we do obtain a column of zeros. Hence the determinant is zero.

Solution to Exercise 42

(a) We have

$$\begin{aligned}\mathbf{r}\times\mathbf{s} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2 & 5 & 0\\ 2 & -1 & 2\end{vmatrix} &= \begin{vmatrix}5 & 0\\ -1 & 2\end{vmatrix}\mathbf{i} - \begin{vmatrix}2 & 0\\ 2 & 2\end{vmatrix}\mathbf{j} + \begin{vmatrix}2 & 5\\ 2 & -1\end{vmatrix}\mathbf{k}\\ &= 10\mathbf{i} - 4\mathbf{j} - 12\mathbf{k} = (10, -4, -12).\end{aligned}$$

(b) The area is given by $|\mathbf{r}_1 \times \mathbf{r}_2|$, where

$$\mathbf{r}_1 \times \mathbf{r}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 2 & 2 & -2 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 2 & -2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 0 \\ 2 & -2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix}\mathbf{k}$$
$$= -2\mathbf{i} + 2\mathbf{j} = (-2, 2, 0).$$

So, taking the magnitude, the area is

$$|\mathbf{r}_1 \times \mathbf{r}_2| = \sqrt{(-2)^2 + (-2)^2} = \sqrt{8} = 2\sqrt{2}.$$

(c) The volume is given by $|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$, where the sides may be taken in any order. We have

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} 2 & 5 & 0 \\ 1 & -2 & 0 \\ 1 & -3 & 2 \end{vmatrix} = 2\begin{vmatrix} -2 & 0 \\ -3 & 2 \end{vmatrix} - 5\begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} + 0\begin{vmatrix} 1 & -2 \\ 1 & -3 \end{vmatrix}$$
$$= 2(-4) - 5(2) = -18.$$

So, taking the magnitude, the volume is

$$|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})| = 18.$$

Note that the determinant may be obtained more quickly by expanding on the third column rather than the first row, in which case we would get

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 0 - 0 + 2\begin{vmatrix} 2 & 5 \\ 1 & -2 \end{vmatrix} = 2(-4 - 5) = -18,$$

as before.

Solution to Exercise 43

The cofactors of $\mathbf{A} = \begin{bmatrix} 2 & -1 & 2 \\ 5 & 1 & -1 \\ 2 & 1 & -1 \end{bmatrix}$ are

$$C_{11} = +\begin{vmatrix} 1 & -1 \\ 1 & -1 \end{vmatrix} = 0, \quad C_{12} = -\begin{vmatrix} 5 & -1 \\ 2 & -1 \end{vmatrix} = 3, \quad C_{13} = +\begin{vmatrix} 5 & 1 \\ 2 & 1 \end{vmatrix} = 3,$$
$$C_{21} = -\begin{vmatrix} -1 & 2 \\ 1 & -1 \end{vmatrix} = 1, \quad C_{22} = +\begin{vmatrix} 2 & 2 \\ 2 & -1 \end{vmatrix} = -6, \quad C_{23} = -\begin{vmatrix} 2 & -1 \\ 2 & 1 \end{vmatrix} = -4,$$
$$C_{31} = +\begin{vmatrix} -1 & 2 \\ 1 & -1 \end{vmatrix} = -1, \quad C_{32} = -\begin{vmatrix} 2 & 2 \\ 5 & -1 \end{vmatrix} = 12, \quad C_{33} = +\begin{vmatrix} 2 & -1 \\ 5 & 1 \end{vmatrix} = 7.$$

Thus

$$\mathbf{C} = \begin{bmatrix} 0 & 3 & 3 \\ 1 & -6 & -4 \\ -1 & 12 & 7 \end{bmatrix} \quad \text{and} \quad \mathbf{C}^T = \begin{bmatrix} 0 & 1 & -1 \\ 3 & -6 & 12 \\ 3 & -4 & 7 \end{bmatrix}.$$

So

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}}\mathbf{C}^T = \frac{1}{3}\begin{bmatrix} 0 & 1 & -1 \\ 3 & -6 & 12 \\ 3 & -4 & 7 \end{bmatrix}.$$

As usual, you should check that $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$ by explicit multiplication.

Acknowledgements

Grateful acknowledgement is made to the following source:

Figure 26: Taken from
https://lhc-div-mms-web.cern.ch/lhc-div-mms/Interconnect.

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